Mathematical Analysis I: Lecture 46

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Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 10:00–11:30.
- Office hour: Tuesday 11:30–12:30.

- Basic Mathematics: first few lessons on
  - Tuesday (14:00 – 16:00 CET): Inequalities, Limits and Derivatives
  - Wednesday (14:00 – 16:00 CET): Study of function

  and then upon request.
- A make up session on 22 December 11:30

- Today: Apostol Vol. 1, Chapter 8

Series with general terms

Let $\{a_n\}$ be a sequence, not necessarily positive. We say that the series $\sum_n a_n$ **converges absolutely** if $\sum_n |a_n|$ converges.

Lemma

If $\sum_n |a_n|$ converges, then so does $\sum_n a_n$.

Proof.

As $\sum_n |a_n|$ converges, for $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for $m > n > N$ it holds that $\sum_{k=n}^m |a_n| < \epsilon$. Then it holds that $|\sum_{k=n}^m a_k| < \epsilon$ by the triangle inequality. This means that $\sum_{k=1}^n a_k$ is a Cauchy sequence, hence it converges. $\square$

This Lemma, combined with the criteria for positive series, enables us to show convergence of many series.

Example

- $\sum_{n=1}^{\infty} \frac{1}{(-1)^n n^s}$ converges absolutely if $s > 1$. Indeed, with $a_n = \frac{1}{(-1)^n n^s}$, $|a_n| = \frac{1}{n^s}$, and we know that $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges if and only if $s > 1$.
- $\sum_{n=1}^{\infty} \frac{n}{(-1)^n 2^n}$ converges absolutely. Indeed, with $a_n = \frac{n}{(-1)^n 2^n}$, $|a_n| = \frac{n}{2^n}$ and by root test,

$$\left(\frac{n}{2^n}\right)^{\frac{1}{n}} = \frac{n^{\frac{1}{n}}}{2} \to \frac{1}{2} < 1.$$

Therefore, $\sum_n |a_n|$ converges.

Series with general terms

If a series $\sum_n a_n$ converges absolutely, its limit does not depend on the order: indeed, as it is absolutely convergent, then its positive elements $b_n = \begin{cases} a_n & \text{if } a_n > 0 \\ 0 & \text{if } a_n \leq 0 \end{cases}$ and the negative elements $c_n = a_n - b_n$ are both convergent. Therefore, even if we sum first the positive elements $\sum_n b_n$ and then the negative elements $\sum_n c_n$, the result is the same:
$\sum_n b_n + \sum_n c_n = \sum_n a_n$.
On the other hand, if a series $\sum_n a_n$ converges but not absolutely, their positive and negative parts both diverges (because otherwise it would be absolute convergence). By rearranging the sum, one can make it diverge to ∞ (by taking many elements of b_n) or to $-\infty$ (by taking many elements of c_n).

A series whose terms change sign at each stem is called an **alternating series**. That is, for $a_n > 0$, it is given by

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots,$$

Lemma (Leibniz criterion)

Let $\{a_n\}$ be a decreasing sequence with positive terms and assume that $a_n \to 0$. Then $\sum(-1)^{n-1}a_n$ converges.

Proof.

Let $s_n = \sum_{k=1}^n (-1)^k a_k$. Then $s_{2n} = \sum_{k=1}^n (-a_{2k-1} + a_{2k})$ is decreasing. Analogously $s_{2n+1} = -a_1 + \sum_{k=1}^n (a_{2k} - a_{2k+1})$ is increasing. In addition, $s_{2n} - s_{2n+1} = -(-a_{2n+1}) = a_{2n+1} \geq 0$. Hence s_{2n} and s_{2n+1} converge to $\overline{s}$ and $\underline{s}$, respectively, and $\overline{s} \geq \underline{s}$. But $s_{2n+1} \leq \underline{s} \leq \overline{s} \leq s_{2n}$ and $s_{2n} - s_{2n+1} = a_{2n+1} \to 0$, hence $\overline{s} = \underline{s}$. $\square$

Series with general terms

Example

- $\sum_n (-1)^{n-1}\frac{1}{n}$ is convergent. Note that this series is not absolutely convergent, indeed, $\sum_n \frac{1}{n}$ is divergent.
- $\sum_n (-1)^{n-1}\frac{1}{\log(n+1)}$ is convergent. Note that this series is not absolutely convergent, indeed, $\sum_n \frac{1}{\log(n+1)}$ is divergent.
- The sequence $\sum_{k=1}^n \frac{1}{k} - \log n$ converges. Indeed, this can be seen as

$$
\begin{aligned}
&1+\frac{1}{2}+\frac{1}{3}\cdots+\frac{1}{n}-\int_1^2 \frac{1}{x}dx-\int_2^3 \frac{1}{x}dx+\cdots-\int_{n-1}^n \frac{1}{x}dx \\
&=\frac{1}{1}-\int_1^2 \frac{1}{x}dx+\frac{1}{2}-\int_2^3 \frac{1}{x}dx+\cdots+\frac{1}{n}-\int_{n-1}^n \frac{1}{x}dx
\end{aligned}
$$

and the last expression is an alternating series. Note that $\frac{1}{k} > \int_k^{k+1}\frac{1}{x}dx > \frac{1}{k+1}$. Furthermore, $\frac{1}{k}\to 0$ as $k\to\infty$. Therefore, the Leibniz criterion applies and the series converges.

Lemma (Abel's partial summation formula)

Let $\{a_n\}, \{b_n\}$ *be two sequences, and let* $A_n = \sum_{k=1}^{n} a_k$. *Then we have the identity* $\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1})$.

Proof.

Let us define $A_0 = 0$. By computing the right-hand side,

$$
\begin{aligned}
A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1}) &= A_n b_{n+1} + \sum_{k=1}^{n} A_k b_k - \sum_{k=2}^{n+1} A_{k-1} b_k \\
&= A_n b_{n+1} + \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n+1} A_{k-1} b_k \\
&= A_n b_{n+1} + \sum_{k=1}^{n} a_k b_k - A_n b_{n+1} \\
&= \sum_{k=1}^{n} a_k b_k .
\end{aligned}
$$

□

Theorem (Dirichlet's test)

Let $\sum_n a_n$ be a series and assume that $A_n = \sum_{k=1}^n a_k$ is a bounded sequence. Let $b_n > 0$ be a decreasing sequence and $b_n \to 0$. Then the series $\sum_n a_n b_n$ converges.

Proof.

By previous Lemma, we have $\sum_{k=1}^n a_n b_n = A_n b_{n+1} + \sum_{k=1}^n A_k(b_k - b_{k+1})$. As b_k is decreasing, we have $b_k - b_{k+1} > 0$ and since A_k is bounded, say by C, we have $\sum_{k=1}^n |A_k(b_k - b_{k+1})| \leq \sum_{k=1}^n C(b_k - b_{k+1})$. The latter is a telescopic series and it is equal to $C(b_1 - b_{n+1})$, which converges (to Cb_1). Therefore, $\sum_{k=1}^n A_k(b_k - b_{k+1})$ is absolutely convergent.
On the other hand, $A_n b_{n+1}$ tends to 0 because $|A_n| < C$ and $b_{n+1} \to 0$.
Altogether, the series $\sum_{k=1}^n a_n b_n$ converges. □

Series with general terms

Theorem (Abel's test)

Let $\sum_n a_n$ be a convergent series and assume b_n is monotonic and convergent. Then the series $\sum_n a_n b_n$ converges.

Proof.

We may assume that b_n is decreasing (otherwise we can consider $-b_n$). In this case, $A_n = \sum_n a_n$ is bounded (because it is convergent), and with $B = \lim b_n$, $b_n - B$ is decreasing and converges to 0.
Hence we can apply Dirichlet's test to the first term of $\sum_n a_n b_n = \sum_n a_n(b_n - B) + \sum_n a_n B$, where the last term is convergent because $\sum_n a_n$ is convergent. □

Series with general terms

Example

Let us consider the series $\sum_n \frac{\sin n\pi\theta}{n^s}$, where $\theta = \frac{p}{q}$ is a rational number and $s > 0$. This is not absolutely convergent, nor alternating (if $q \neq 2$). On the other hand, the sequence $\sin n\pi\theta$ is periodic, because $\sin x$ is 2π-periodic, that is, $\sin \frac{n\pi p}{q} = \sin \frac{(n+2q)\pi p}{q}$. Let us assume q is even and p is odd. Then $\sin \frac{(n+q)\pi p}{q} = -\sin \frac{n\pi p}{q}$, and hence also the sum $\sum_n \sin n\pi\theta$ is periodic, and in particular bounded.

In this case, as $\frac{1}{n^s}$ is monotonically decreasing and tends to 0, we can apply Dirichlet's test and conclude that $\sum_{n=1}^{\infty} \frac{\sin n\pi\theta}{n^s}$ is convergent.

Similarly, $\sum_{n=1}^{\infty} \frac{\cos n\pi\theta}{n^s}$

The same holds for q odd, and actually for any $\theta \in \mathbb{R}$.

Series with general terms

- For $x \in \mathbb{R}$, consider $\sum_{n=1}^{\infty} \frac{x^n}{n2^n}$. This converges for $|x| < 2$ absolutely, and diverges for $|x| > 2$, and diverges for $x = 2$ and converges for $x = -2$.
- For $x \in \mathbb{R}$, consider $\sum_{n=1}^{\infty} \frac{x^n}{n!}$. This converges for all x absolutely.

Exercises

- Determine whether $\sum_n \frac{(-1)^n n}{n^2+1}$ converges.
- Determine whether $\sum_n \frac{(-1)^n}{n \log n}$ converges.
- Determine for which x $\sum_n \frac{x^{2n}}{x^{2n}+1}$ converges.
- Determine for which x $\sum_n \frac{n^2 x^{2n}}{(2n)!}$ converges.
- Determine for which x $\sum_n \frac{(-1)^n x^{2n}}{n}$ converges.