Mathematical Analysis I: Lecture 45

Lecturer: Yoh Tanimoto

09/12/2020

Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 10:00–11:30.
- Office hour: Tuesday 11:30–12:30.

- Basic Mathematics: first few lessons on
  - Tuesday (14:00 – 16:00 CET): Inequalities, Limits and Derivatives
  - Wednesday (14:00 – 16:00 CET): Study of function

  and then upon request.
- A make up session on 22 December 11:30

- Today: Apostol Vol. 1, Chapter 8

Series with positive terms

Let$\{a_n\} \subset \mathbb{R}$ be a sequence. We have considered a series $\sum_{k=1}^{n} a_k$, which is a new sequence and its convergence or divergence.

When all the term are non-negative: $a_n \geq 0$, there are some criteria that can be often used to determine the convergence or divergence.

Series with positive terms

Theorem

We have the following.

- *Let* $\{a_n\}, \{b_n\}$ *be two sequences,* $a_n \geq 0, b_n \geq 0$ *such that there is* $c > 0$ *and* $a_n \leq cb_n$ *for n sufficiently large. In this case, if* $\sum b_n$ *converges, then so does* $\sum a_n$*. If* $\sum a_n$ *diverges, so does* $\sum b_n$*.*
- *Let* $\{a_n\}, \{b_n\}$ *be two sequences,* $a_n \geq 0, b_n \geq 0$ *such that* $a_n/b_n \to c, c \neq 0, \infty$*. Then* $\sum a_n$ *converges if and only if* $\sum b_n$ *converges.*

Proof.

- As $a_n \geq 0, b_n \geq 0$, the series $\sum_n a_n, \sum_n b_n$ are increasing. If $\sum_n b_n$ is convergent and $a_n \leq cb_n$, then $\sum_{k=1}^n a_k \leq \sum_{k=1}^n cb_k \leq \sum_{k=1}^\infty cb_k$, hence the former is bounded and increasing, therefore, it must converge. The other case is analogous.
- If $a_n/b_n \to c \neq 0$, then this implies that $\frac{c}{2}b_n \leq a_n \leq 2cb_n$ for sufficiently large n, hence the previous point applies.

□

Series with positive terms

We have seen that, for $0 < a < 1$, $\sum_{k=1}^{n} a^k$ converges to $\frac{1}{1-a}$. We can use this fact to show the convergence of some other series.

Theorem (root test)

Let $a_n > 0$ be a sequence.

1. *If $a_n^{\frac{1}{n}} \le \theta < 1$ for n sufficiently large, then $\sum_n a_n$ converges.*
2. *If $a_n^{\frac{1}{n}} \ge \theta > 1$ for n sufficiently large, then $\sum_n a_n$ diverges.*
3. *Let $a_n^{\frac{1}{n}} \to a$. If $a < 1$, $\sum_n a_n$ converges. If $a > 1$, $\sum_n a_n$ diverges.*

Proof.

The series $\sum_n \theta^n$ converges if $\theta < 1$ (the geometric series) and diverges if $\theta > 1$ (θ^n does not tend to 0). By Theorem 1, and $a_n^{\frac{1}{n}} < \theta$ or $a_n^{\frac{1}{n}} > \theta$, the first two claims follow.
If $a_n^{\frac{1}{n}} \to a < 1$, then we can take θ such that $a_n^{\frac{1}{n}} < \theta < 1$ for n sufficiently large. The case $a_n^{\frac{1}{n}} \to a > 1$ is analogous. □

Example

- $\sum_n \frac{1}{n3^n}$ is convergent.
- $\sum_n \frac{n2^n}{3^n}$ is convergent.
- $\sum_n \frac{n}{2^n}$ is convergent.

When $\lim_n a_n^{\frac{1}{n}} = 1$, this criterion does not give information. Indeed, $\sum_n \frac{1}{n}$ is divergent, but $\sum_n \frac{1}{n^2}$ is convergent (compare it with $\sum \frac{1}{n(n-1)}$), while in both cases $\lim(\frac{1}{n})^{\frac{1}{n}} = \lim(\frac{1}{n^2})^{\frac{1}{n}} = 1$.

Theorem (ratio test)

Let $a_n > 0$ be a sequence.

1. *If $\frac{a_{n+1}}{a_n} \leq \theta < 1$ for n sufficiently large, then $\sum_n a_n$ converges.*
2. *If $\frac{a_{n+1}}{a_n} \geq \theta > 1$ for n sufficiently large, then $\sum_n a_n$ diverges.*
3. *Let $\frac{a_{n+1}}{a_n} \to a$. If $a < 1$, $\sum_n a_n$ converges. If $a > 1$, $\sum_n a_n$ diverges.*

Proof.

Let $\frac{a_{n+1}}{a_n} \leq \theta < 1$ for $n \geq N$. Then,

$$a_{N+m} \leq a_{N+m-1}\theta \leq a_{N+m-2}\theta^2 \leq \cdots \leq a_N\theta^m.$$

Now $\sum_n a_N\theta^m$ is convergent, hence by Theorem 1, $\sum_n a_n$ is convergent. If $\frac{a_{n+1}}{a_n} \geq \theta > 1$, then a_n is increasing and does not convergent to 0.
If $\frac{a_{n+1}}{a_n} \to a < 1$ or > 1, then $\frac{a_{n+1}}{a_n} \leq \theta < 1$ or > 1 for n sufficiently large, hence the claim follow from (i), (ii). □

Series with positive terms

Example

- $\sum \frac{n}{2^n}$ is convergent.
- $\sum \frac{n^2}{n!}$ is convergent.
- $\sum \frac{(n!)^2}{2^{n^2}}$ is convergent.

When $a_{n+1}/a_n \to 1$ or $a_n^{\frac{1}{n}} \to 1$, we need to study the series better.

Series with positive terms

Lemma (integral test)

Let $\{a_n\}$ be a decreasing sequence of positive numbers and assume that there is a positive decreasing function $f(x)$ defined on $[1, \infty)$. If $a_n \leq f(x)$ and $\int_1^\infty f(x)dx$ converges, then $\sum_n a_n$ converges. If $a_n \geq f(x)$ and $\int_1^\infty f(x)dx$ diverges, then $\sum_n a_n$ diverges.

Proof.

For the first case, we have $\sum_{k=2}^n a_n \leq \int_1^n f(x)dx$, and the later converges, hence so does the former.
For the first case, we have $\sum_{k=1}^n a_n \geq \int_1^{n+1} f(x)dx$, and the later diverges, hence so does the former. □

Series with positive terms

Example

Let us fix $s \in \mathbb{R}$ and consider $\sum_{n=1}^{\infty} \frac{1}{n^s}$. We can compare this with $f_s(x) = \frac{1}{n^s}$. We know that $\int_1^{\infty} f_s(x)dx$ converges if and only if $s > 1$. $\zeta(s)$ is called the Riemann zeta function.

Series with positive terms

Lemma (condensation principle)

Let $\{a_n\}$ *be a decreasing sequence of positive numbers. Then* $\sum a_n$ *converges if and only if* $\sum 2^n a_{2^n}$.

Proof.

Since a_n is decreasing and positive, $a_{2^n} \geq a_{2^n+1} \geq \cdots \geq a_{2^{n+1}}$, hence

$$2^n a_{2^n} \geq \sum_{k=2^n}^{2^{n+1}-1} a_k \geq 2^n a_{2^{n+1}}.$$

By summing this with respect to n,

$$\sum_{n=1}^{N} 2^n a_{2^n} \geq \sum_{n=1}^{N} \sum_{k=2^n}^{2^{n+1}-1} a_k = \sum_{n=1}^{2^{N+1}-1} a_k$$
$$\geq \sum_{n=1}^{N} 2^n a_{2^{n+1}} = \frac{1}{2} \sum_{n=1}^{N} 2^{n+1} a_{2^{n+1}} = \frac{1}{2} \left(\sum_{n=1}^{N+1} 2^n a_{2^n} - a_1 \right).$$

Therefore, $\sum_n a_n$ converges if and only if $\sum_n 2^n a_{2^n}$ converges by Theorem 1. □

Series with positive terms

Example

$\sum \frac{1}{n(\log n)^\alpha}$. By condensation principle, it is enough to study $\sum 2^n \frac{1}{2^n(\log 2^n)^\alpha} = \sum \frac{1}{(n\log 2)^\alpha} = \frac{1}{\log 2^\alpha} \sum \frac{1}{n^\alpha}$. Hence this converges if and only if $\alpha > 1$.

Exercises

- Determine whether $\sum_n \frac{n}{n^2+1}$ converges.
- Determine whether $\sum_n \frac{(n!)^2}{(2n)!}$ converges.
- Determine whether $\sum_n \frac{2^n+3^n}{5^n}$ converges.
- Determine whether $\sum_n \frac{\log n}{n^2}$ converges.
- Determine whether $\sum_n \frac{1}{(\log x)^2}$ converges.