Mathematical Analysis I: Lecture 44

Lecturer: Yoh Tanimoto

04/12/2020

Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 10:00–11:30.
- Office hour: Tuesday 11:30–12:30.

- Basic Mathematics: first few lessons on
  - Tuesday (14:00 – 16:00 CET): Inequalities, Limits and Derivatives
  - Wednesday (14:00 – 16:00 CET): Study of function

  and then upon request.
- **No Lecture/no tutoring on 7, 8 December**.
- A make up session on 22 December 11:30

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$$\int_0^\beta x^{\frac{1}{3}} e^{-x^{\frac{4}{3}}} dx = -\frac{3}{4}\left[e^{-x^{\frac{4}{3}}}\right]_0^\beta = -\frac{3}{4}(e^{-\beta^{\frac{4}{3}}} - e^0)$$

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By taking the limit $\beta \to \infty$, we have $\int_0^\infty x^{\frac{1}{3}} e^{-x^{\frac{4}{3}}}\,dx = \frac{3}{4}$.

Calculate the following improper integral. $\int_1^{\infty} \frac{\log x}{x^2} dx$

Solution. The function $\frac{\log x}{x^2}$ is bounded on any interval of the form $[1, \beta]$. The integral is improper only as $x \to \infty$.

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$$\int_1^\beta \frac{\log x}{x^2}dx = \left[-\frac{\log x}{x}\right]_1^\beta + \int_1^\beta \frac{1}{x^2}dx$$

$$= -\frac{\log \beta}{\beta} + 0 + \left[-\frac{1}{x}\right]_1^\beta = -\frac{\log \beta}{\beta} + (-\frac{1}{\beta} - (-1))$$

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By taking the limit $\beta \to \infty$, we have $\int_1^\beta \frac{\log x}{x^2} dx = 1$.

Determine whether the following improper integral converges. $\int_1^\infty \frac{x^3}{x^4+1}dx$.

Solution 1. The function $\frac{x^3}{x^4+1}$ is bounded on any interval of the form $[1, \beta]$. The integral is improper only as $x \to \infty$.

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Solution 1. The function $\frac{x^3}{x^4+1}$ is bounded on any interval of the form $[1, \beta]$. The integral is improper only as $x \to \infty$.Furthermore, $\frac{x^3}{x^4+1}$ is asymptotically equal to $\frac{1}{x}$ as $x \to \infty$, that is,

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On the other hand, we know that $\int_1^\beta \frac{1}{x}dx = [\log x]_0^\beta = \log\beta$ diverges as $\beta\to\infty$. Therefore, the integral $\int_1^\infty \frac{x^3}{x^4+1}dx$ diverges as well.

Solution 2. $\int_1^\infty \frac{x^3}{x^4+1}dx = \frac{1}{4}[\log(x^4+1)]_1^\beta = \frac{1}{4}(\log(\beta^4+1)-\log 2)$ and this diverges as $\beta\to\infty$.

Determine whether the following improper integral converges.
$\int_1^2 \frac{x^2}{(x-1)^{\frac{1}{2}}} dx$.

Solution. The function $\frac{x^2}{(x-1)^{\frac{1}{2}}}$ is bounded on any interval of the form $[1+\epsilon, 2]$ for $\epsilon > 0$. The integral is improper only as $x \to 1$.

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On the other hand, we know that
$\int_{1+\epsilon}^2 \frac{1}{(x-1)^{\frac{1}{2}}} dx = [2(x-1)^{\frac{1}{2}}]_{1+\epsilon}^2 = 2 - 2\epsilon^{\frac{1}{2}}$ converges (to 2 as $\epsilon \to 0$) as $\epsilon \to 0$. Therefore, the improper integral $\frac{x^2}{(x-1)^{\frac{1}{2}}}$ is also convergent.

Determine whether the following improper integral converges. $\int_0^1 \frac{x^2}{\sin x - x} dx$.

Solution. The function $\frac{x^2}{\sin x - x}$ is bounded on any interval of the form $[\epsilon, 1]$ for $\epsilon > 0$. The integral is improper only as $x \to 0$.

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Solution. The function $\frac{x^2}{\sin x - x}$ is bounded on any interval of the form $[\epsilon, 1]$ for $\epsilon > 0$. The integral is improper only as $x \to 0$.Furthermore, as $\sin x - x = -\frac{x^3}{6} + o(x^3)$, $\frac{x^2}{\sin x - x}$ is asymptotically equal to $\frac{x^2}{\frac{x^3}{6}}$ as $x \to 0$, that is,

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On the other hand, we know that $\int_\epsilon^1 \frac{x^2}{-\frac{x^3}{6}} dx = -6[\log x]_\epsilon^1 = 6 \log \epsilon$ diverges as $\epsilon \to 0$. Therefore, the improper integral $\frac{x^2}{\sin x - x}$ is also divergent.

Calculate the area of the region surrounded by $y = x^2 - 1$ and the x-axis.
Solution. The function $y = x^2 - 1$ and the x-axis intersects when $x^2 - 1 = 0$, that is, at $x = -1, 1$.

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$$\int_{-1}^{1} 0 - (x^2 - 1)dx = [x - \frac{x^3}{3}]_{-1}^{1} = \frac{4}{3}.$$

Calculate the area of the region surrounded by $y = x^2$ and $y = 5x + 6$.
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$$\int_{-1}^{6} (5x+6-x^2)dx = [\frac{5x^2}{2}+6x-\frac{x^3}{3}]_{-1}^{6} = (90+36-72)-(\frac{5}{2}-6-\frac{1}{3}) = \frac{343}{6}.$$

Calculate the area of the region given by $\{(x, y) : a^2x^2 + b^2y^2 \leq 1\}$.
Solution. The condiction can be written equivalently as $b^2y^2 \leq 1 - a^2x^2$,

$$-\frac{1}{b}\sqrt{1 - a^2x^2} \leq y \leq \frac{1}{b}\sqrt{1 - a^2x^2}.$$

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The area is given, with the change of variables $x = \frac{t}{a}$ and $\frac{dx}{dt} = \frac{1}{a}$, $t = \sin\theta, \frac{dt}{d\theta} = \cos\theta$, by

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$$\begin{aligned}
&\int_{-\frac{1}{a}}^{\frac{1}{a}} \frac{1}{b}\sqrt{1-a^2x^2} - (-\frac{1}{b}\sqrt{1-a^2x^2})dx \\
&= \frac{2}{b}\int_{-\frac{1}{a}}^{\frac{1}{a}} \sqrt{1-a^2x^2}dx = \frac{2}{b}\int_{-1}^{1} \sqrt{1-t^2}\frac{1}{a}dt = \frac{2}{ab}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta d\theta \\
&= \frac{2}{ab}[\frac{\cos 2\theta + 1}{2}]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2\pi}{ab}.
\end{aligned}$$

Calculate the length of the curve given by $f(x) = \frac{x^2}{2}$ from $x = -\frac{e-\frac{1}{e}}{2}$ to $x = \frac{e-\frac{1}{e}}{2}$.

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$$\begin{aligned}
&\int_{-\frac{e-\frac{1}{e}}{2}}^{\frac{e-\frac{1}{e}}{2}} \sqrt{1+x^2}dx \\
&= \int_{-1}^{1} \cosh^2 t dt = \int_{-1}^{1} \frac{e^{2x} + 2 + e^{-2x}}{2} dt \\
&= \frac{1}{4}[e^{2x} + 2x - e^{2x}]_{-1}^{1} = 1 + \frac{e^2 - e^{-2}}{2}.
\end{aligned}$$

This and the rest will be done on next Friday!

Compute the series $\sum_{n=1}^{\infty} \frac{3}{4^n}$.
Solution. Note that

Compute the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$.
Solution.

Compute the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$.
Solution.

Compute the series $\sum_{n=1}^{\infty} \frac{1+n}{n!}$.
Solution.