Mathematical Analysis I: Lecture 43

Lecturer: Yoh Tanimoto

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Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 10:00–11:30.
- Office hour: Tuesday 11:30–12:30.

- Basic Mathematics: first few lessons on
  - Tuesday (14:00 – 16:00 CET): Inequalities, Limits and Derivatives
  - Wednesday (14:00 – 16:00 CET): Study of function

  and then upon request.
- **No Lecture/no tutoring on 7, 8 December**.
- A make up session on 22 Dicember?

- Today: Apostol Vol. 1, Chapter 10

Zeno's paradox

The Achilles and Tortoise paradox goes as follows: Achilles (ancient Greek hero, runs very fast) running behind a tortoise (walks very slowly). At the beginning, Achilles is 10 meter behind the tortouse. In the next moment, Achilles arrives at the position where the tortouse was there, but in the meantime it moves by 1 meter. Then Achilles arrives at the position where the tortoise was there in the previous moment, but in the meantime it moves by 0.1 meter. Then Achilles arrives...
So how can we be sure that Achilles catches up with the tortoise?

Series

Let us recall that we have considered **sequences** of numbers $a_1, a_2, \cdots, a_n, \cdots$, and the **series** $\sum_{k=1}^n a_k$, that is a new sequence

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \cdots, \sum_{k=1}^{n} a_k, \cdots$$

As this is a new sequence, we can consider its convergence or divergence. That is, we say that a series $\sum_{k=1}^n a_k$ **converges** to $S \in \mathbb{R}$ if for each $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for $n > N$ it holds that $|S - \sum_{k=1}^n a_k| < \epsilon$. We say taht the series **diverges** if for any $R > 0$ there is $N \in \mathbb{N}$ such that for $n > N$ it holds that $|\sum_{k=1}^n a_k| > R$. In other cases we just say that the series does not converge.
If a series converges, we denote the limit by $\sum_{k=1}^\infty a_k$. Sometimes we just write $\sum_n a_n$ a general term in a series.

Series

Example

- $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. This diverges.
- $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. This diverges.
- $\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$ (if $a \neq 1$) This converges if and only if $|a| < 1$.

For these examples, we know the exact form of the n-th sum. For other series, it is difficult to compute such general term, but still we may be able to say whether the series converges or not.

For example, let us take $a_n = \frac{1}{n}$ and consider $\sum_{k=1}^{n} \frac{1}{k}$. This is called the **harmonic series**. As we have seen, this sum is larger than the integral of $\frac{1}{x}$ on $[1, n+1]$:

$$\int_1^{n+1} \frac{1}{x} dx \le \sum_{k=1}^{n} \frac{1}{k}.$$

On the other hand, we can calculate the left-hand side and we obtain $\int_1^{n+1} \frac{1}{x} dx = [\log x]_1^{n+1} = \log(n+1)$, and this diverges as $n \to \infty$. Therefore, $\sum_{k=1}^{n} \frac{1}{k}$ diverges as well.

Series

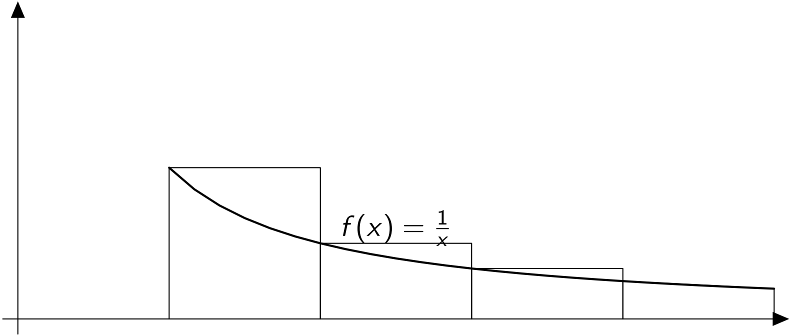

Figure: A graphical proof that $\sum_{n=1}^{N} \frac{1}{n}$ diverges as $N \to \infty$.

Series

Lemma

If $\sum_{k=1}^{\infty} a_k$ *converges, then* $a_k \to 0$ *as* $k \to \infty$*.*

Proof.

As $\sum_{k=1}^{\infty} a_k$ is convergent to S, for any $\epsilon > 0$ there is N such that if $n > N$ then $|\sum_{k=1}^{n} a_k - S| < \frac{\epsilon}{2}$. In particular, we have $S - \frac{\epsilon}{2} < \sum_{k=1}^{n} a_k < S + \frac{\epsilon}{2}$ and $S - \frac{\epsilon}{2} < \sum_{k=1}^{n+1} a_k < S + \frac{\epsilon}{2}$.
From this it follows that $-\epsilon < \sum_{k=1}^{n+1} a_k - \sum_{k=1}^{n} a_k < \epsilon$, that is, $|\sum_{k=1}^{n+1} a_k - \sum_{k=1}^{n+1} a_k| = |a_{n+1}| < \epsilon$. This means that $a_n \to 0$. $\square$

Series

Example

- $\sum_{k=1}^{n} k$

Example

- $\sum_{k=1}^{n} k$ does not converge because $a_k = k$ diverges.

Series

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- $\sum_{k=1}^{n} \left(\frac{1}{2}\right)^k$

Series

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- $\sum_{k=1}^{n} (\frac{1}{2})^k$ converges to 1, and in this case indeed $(\frac{1}{2})^k$ converges to 0.

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- $\sum_{k=1}^{n} \frac{1}{k(k+1)}$

Example

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- $\sum_{k=1}^{n} \frac{1}{k}$ diverges, although in this case indeed $\frac{1}{k}$ converges to 0.
- $\sum_{k=1}^{n} \frac{1}{k(k+1)}$ converges, as we will see later.

Theorem

(i) *Let* $\sum_{k=1}^{\infty} a_k$ *be convergent. Then for any* $c \in \mathbb{R}$, $\sum_{k=1}^{\infty} ca_k$ *is also convergent and* $\sum_{k=1}^{\infty} ca_k = c\sum_{k=1}^{\infty} a_k$.

(ii) *Let* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *be convergent. Then* $\sum_{k=1}^{\infty}(a_k + b_k)$ *is also convergent and* $\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty}(a_k + b_k)$

(iii) *Let* $\sum_{k=1}^{\infty} a_k$ *convergent and* $\sum_{k=1}^{\infty} b_k$ *be divergent. Then* $\sum_{k=1}^{\infty}(a_k + b_k)$ *is divergent.*

Proof.

(i)(ii) These follow from the properties of sequences.

(iii) Suppose the contrary that $\sum_{k=1}^{\infty}(a_k + b_k)$ converges. Then by (ii) $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty}(a_k + b_k) - a_k$ would converge, which contradicts the assumption that $\sum_{k=1}^{\infty} a_k + b_k$ diverges.

□

Series

Example

- The series $\sum_n(\frac{1}{n} + \frac{1}{2^n})$ diverges, because $\sum_n \frac{1}{n}$ diverges while $\sum_n \frac{1}{2^n}$ converges.
- The series $\sum_n 1$ and $\sum_n -1$ both diverge, but the sum $\sum_n(1-1) = \sum_n 0$ converges to 0.

Let us consider some cases where the sum conveges.

Example

- Telescopic series. Let b_n a sequence and $a_n = b_{n+1} - b_n$. We call such $\sum_n a_n$ a **telescopic series**. (Any series can be written in the form of telescopic series, but we are interested in the case where b_n is simpler than a_n)
  Then we have
  $$\sum_{k=1}^{n} a_n = (b_2 - b_1) + (b_3 - b_2) + \cdots + (b_{n+1} - b_n) = b_{n+1} - b_1.$$
  From this we infer that $\sum_n a_n$ is convergent if and only if b_n is convergent.

Example

- For example, consider $a_n = \frac{1}{n(n+1)}$. This can be written as
$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$
hence with $b_n = -\frac{1}{n}$, this is a telescopic series. By the argument above, we see that $\sum_{k=1}^{n} \frac{1}{n(n+1)} = b_{n+1} - b_1 = 1 - \frac{1}{n+1}$, and $\sum_{k=1}^{\infty} a_n = 1$.
Next let us take $a_n = \log(n/(n+1))$. Then it holds that $a_n = \log n - \log(n+1)$, hence with $b_n = -\log n$ this is a telescopic series. As $b_n \to -\infty$, the series $\sum_n a_n$ diverges.

Example

- Geometric series. Let us take $x \in \mathbb{R}, x \neq 1$. We know that $\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$. It is clear that the series converges to $\frac{1}{1-a}$ if $|a| < 1$, and diverges if $|a| > 1$. If $a = 1$, then the series is simply $\sum_{k=1}^{n} 1 = n + 1$ and diverges as well.

Series

Geometric series can be seen as a function of x: For a given number $x \in \mathbb{R}$, we consider a sequence $a_n(x) = x^n$ and it holds for $|x| < 1$ that

$$\sum_{n=0}^{\infty} a_n(x) = \frac{1}{1-x}.$$

The right-hand side is again a function of x. In this sense, a convergent series which depends on x defines a new function.

Series

We have seen other examples of this type:

$$e^x = \sum_{n=0} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} \cdots$$

$$\sin x = \sum_{n=0} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots$$

$$\cos x = \sum_{n=0} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots$$

In all these cases, for a fixed $x \in \mathbb{R}$, we have seen that the right-hand converges by the Taylor formula with remainder.
In a similar way, we can define many other useful functions by series.

Exercises

- Compute the series $\sum_{n=1}^{\infty} \frac{3}{4^n}$.
- Compute the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$.
- Compute the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$.
- Compute the series $\sum_{n=1}^{\infty} \frac{1+n}{n!}$.