Mathematical Analysis I: Lecture 42

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Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 10:00–11:30.
- Office hour: Tuesday 11:30–12:30.

- Basic Mathematics: first few lessons on
  - Tuesday (14:00 – 16:00 CET): Inequalities, Limits and Derivatives
  - Wednesday (14:00 – 16:00 CET): Study of function

  and then upon request.
- **No Lecture/no tutoring on 7, 8 December**

- Today: Apostol Vol. 1, Chapter 1

Area

We know the area of rectangles, triangles and disks Let us define the area of a more general region.

Definition

- Let $f \geq g$ be two integrable functions on an interval I. The area of the region between g, f is defined by the following:

$$D_{g,f} := \{(x, y) \in \mathbb{R}^2 : x \in I, g(x) \leq y \leq f(x)\}$$
$$\mathrm{area}(D_{g,f}) := \int_I (f(x) - g(x))dx.$$

- Even if I is not bounded, if the improper integral $\int_I (f(x) - g(x))dx$ exists, then we define the area of the region $D_{g,f} = \{(x, y) \in \mathbb{R}^2 : x \in I, g(x) \leq y \leq f(x)\}$ by the same formula.
- If D is the disjoint union of such regions, then $\mathrm{area}(D)$ is the sum of the areas of such regions.

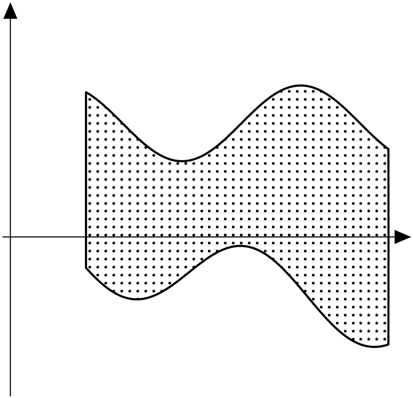

Figure: The area of the region between two functions.

Example

- Rectancles. $D = \{(x,y) \in \mathbb{R}^2 : x \in I, a \le y \le b\}$, with the length $|I|$ and width $b-a$, then $\mathrm{area}(D) = \int_I (b-a)dx = (b-a)|I|$.
- Triangles. $D = \{(x,y) \in \mathbb{R}^2 : x \in [0,a], 0 \le y \le \frac{b}{a}x\}$, with length a and width b, then $\mathrm{area}(D) = \int_0^a \frac{b}{a}x dx = [\frac{b}{2a}x^2]_0^a = \frac{ab}{2}$.

Area

Example

- Disks. $D = \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} \le a\}$, with radius a, then D can be also written as

$$y^2 \le a^2 - x^2 \Longleftrightarrow -\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}.$$

Furthermore, $-a \le x \le a$ because, if $x > a$ then there is no y such that $x^2+y^2 \le a^2$. Therefore,

$$D = \{(x,y) \in \mathbb{R}^2 : -a \le x \le a, -\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}\}$$

for which we can compute the area.

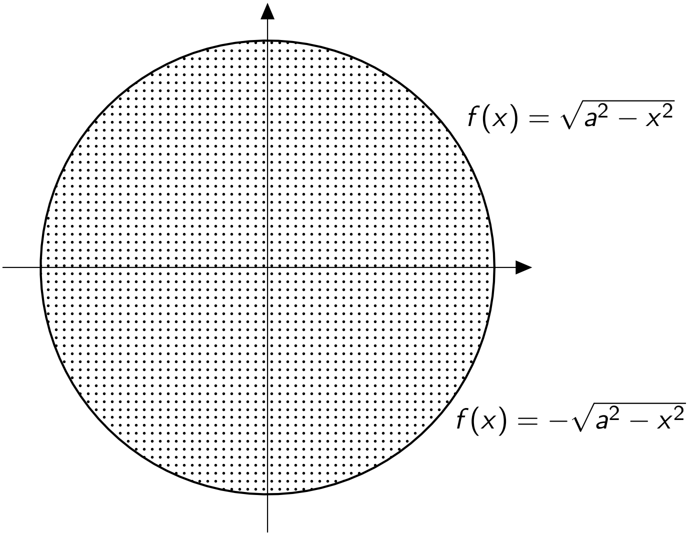

Figure: The area of a disk.

Area

Example

By our definition,

$$\mathrm{area}(D) = \int_{-a}^{a} (\sqrt{a^2 - x^2} - (-\sqrt{a^2 - x^2}))dx$$
$$= 2\int_{-a}^{a} \sqrt{a^2 - x^2}dx.$$

By change of variables $x = a\sin\theta$ with $\frac{dx}{d\theta} = a\cos\theta$, this integral corresponds to that on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ because $a\sin\frac{\pi}{2} = a, a\sin(-\frac{\pi}{2}) = -a$,

$$\int_{-a}^{a} \sqrt{a^2 - x^2}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2\theta}a\cos\theta d\theta.$$

Area

Example

Using $\cos^2\theta = \frac{\cos 2\theta + 1}{2}$,

$$\begin{aligned}
\mathrm{area}(D) &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2\theta}\, a\cos\theta d\theta \\
&= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2\cos^2\theta d\theta \\
&= a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos 2\theta + 1) d\theta \\
&= a^2\left[\frac{\sin 2\theta}{2} + \theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= a^2\pi.
\end{aligned}$$

Length

A curve can be, at least partially, described using a function.

- segment. $\{(x, y) : x \in I, y = ax + b\}$.
- semicircle. $\{(x, y) : x \in (-a, a), y = \sqrt{a^2 - x^2}\}$.
- parabola. $\{(x, y) : x \in \mathbb{R}, y = x^2\}$.
- hyperbola. $\{(x, y) : x \in \mathbb{R}, y = \sqrt{x^2 + 1}\}$.

As we defined the area of a general region using integral, we can define the length of a curve with integral.

Definition

For a curve represented by $G_f := \{(x, y) : x \in [a, b], y = f(x)\}$, where f is differentiable and f' is continuous, we define the length by

$$\ell(G_f) := \int_a^b \sqrt{1 + f'(x)^2}dx.$$

If G_f is a union of such graphs, then $\ell(G_f)$ is defined to be the sum of the lengths of the partial graphs.

Length

Let us see that this coincides with the case of segment: a segment that goes by a horizontally and b vertically is represented by $\{(x,y) : x \in [0,a], y = \frac{b}{a}x\}$. Hence $f(x) = \frac{b}{a}x, f'(x) = \frac{b}{a}$. By definition, $\ell(G_f) = \int_0^a \sqrt{1+(\frac{b}{a})^2}dx = a\sqrt{1+(\frac{b}{a})^2} = \sqrt{a^2+b^2}$, which coincides with the length of the segment by the theorem of Pytagoras.

If a curve is a union of different parts, each of which is represented by a function f_j, then the length of the curve is the sum of the lengths of the parts.

Length

Another possibile definition is to approximate a curve by segments: let $f(x)$ be a function on $I = [a, b]$ and take a parition P by $a = x_0 < x_1 < \cdots < x_n = b$. Correspondingly, we consider the attached segments $P_f(\{x_k\})$: $(x_0, f(x_0)), (x_1, f(x_1)), \cdots, (x_n, f(x_n))$. Let us recall $|P| = \max_{1\le k\le n-1}\{x_{k+1} - x_k\}$.

Length

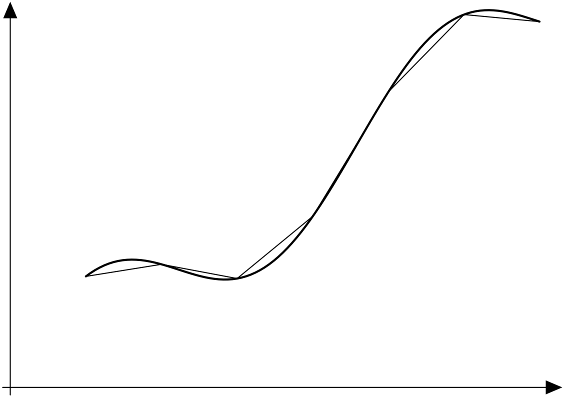

Figure: The length and segments.

Length

Lemma

Let f be differentiable and f' be continuous. Then for any ϵ there is δ such that if $a = x_0 < x_1 < \cdots < x_n = b$ is a partition P with $|P| < \delta$, then $|\ell(G_f) - \ell(P_f(\{x_k\}))| < \epsilon$.

Proof.

By the mean value theorem, there are $x_k \le \xi_k \le x_{k+1}$ such that $f(x_k) - f(x_{k+1}) = f'(\xi_k)(x_k - x_{k+1})$. Since $P_f(\{x_k\})$ is the union of segments,

$$\begin{aligned}
\ell(P_f(\{x_k\}) &= \sum_{k=1}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + (f(x_{k+1}) - f(x_k))^2} \\
&= \sum_{k=1}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + f'(\xi_k)^2 (x_{k+1} - x_k)^2} \\
&= \sum_{k=1}^{n-1} \sqrt{1 + f'(\xi_k)^2}(x_{k+1} - x_k).
\end{aligned}$$

Proof.

On the other hand, $\sqrt{1+f'(x)^2}$ is continous and hence integrable. By uniform continuity, there is δ such that $|\sqrt{1+f'(x)^2} - \sqrt{1+f'(y)^2}| < \frac{\epsilon}{b-a}$ if $|x-y| < \delta$. With such a parition, we have $\underline{S}_I(\sqrt{1+f'(x)^2}, P) \leq \ell(P_f(\{x_k\}) \leq \overline{S}_I(\sqrt{1+f'(x)^2}, P)$.
If $|P|$ is small, the difference between these sides are smaller than ϵ, and $\underline{S}_I(\sqrt{1+f'(x)^2}, P) \leq \ell(G_f) \leq \overline{S}_I(\sqrt{1+f'(x)^2}, P)$. Therefore, $|\ell(G_f) - \ell(P_f(\{x_k\}))| < \epsilon$. □

Length

Example

Semicircle. $I = [-1, 1], f(x) = \sqrt{1-x^2}, f'(x) = \frac{x}{\sqrt{1-x^2}}$.

$$\begin{aligned}\ell(G_f) &= \int_{-1}^{1} \sqrt{1 + f'(x)^2} dx \\ &= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx = [\arcsin x]_{-1}^{1} = \pi\end{aligned}$$

(note that this is an improper integral). That is, the length of the circle is 2π.

Exercises

- Calculate the are of the region surrounded by $y = x^2 - 1$ and the x-axis.
- Calculate the are of the region surrounded by $y = x^2$ and $y = 5x + 6$.
- Calculate the are of the region given by $\{(x, y) : ax^2 + by^2 = 1\}$.
- Calculate the length of the curve given by $f(x) = x^2$ from $x = -\frac{e - \frac{1}{e}}{2}$ to $x = \frac{e - \frac{1}{e}}{2}$.