Mathematical Analysis I: Lecture 41

Lecturer: Yoh Tanimoto

30/11/2020

Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 10:00–11:30.
- Office hour: Tuesday 11:30–12:30.

- Basic Mathematics: first few lessons on
  - Tuesday (14:00 – 16:00 CET): Inequalities, Limits and Derivatives
  - Wednesday (14:00 – 16:00 CET): Study of function

  and then upon request.

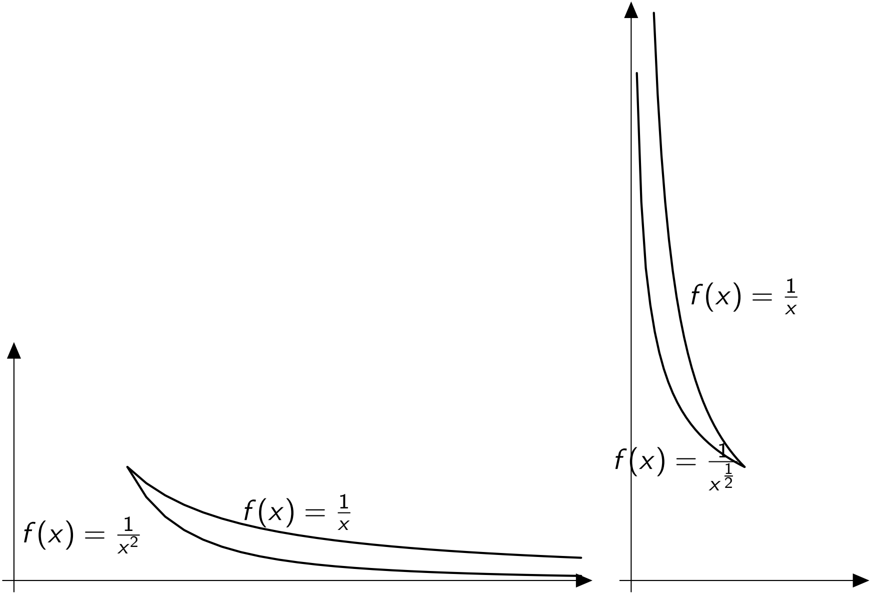
- Today: Apostol Vol. 1, 7.5–,

Improper integral

Let us recall that we introduced a proper integral for an unbounded function or on an unbounded interval by

$$\int_a^\beta f(x)dx = \lim_{\alpha \to a} \int_\alpha^\beta f(x)dx,$$

where $a < \alpha$, and the function is bounded and integrable on all bounded intervals $[\alpha, \beta]$. Similarly, $\int_\alpha^b f(x)dx = \lim_{\beta \to b} \int_\alpha^\beta f(x)dx$ and $\int_a^b f(x)dx = \int_a^{x_0} f(x)dx + \int_{x_0}^b f(x)dx$, where $x_0 \in (a, b)$.
When these limits exist, we say that the improper integral **converges**, and otherwise **does not converge**, or **diverges** if the limit tends to ∞ or $-\infty$.


f(x) = 1/x^2
f(x) = 1/x
f(x) = 1/x
f(x) = 1/x^(1/2)

Improper integral

Example

- The improper integral $\int_0^\infty \sin x dx$

Improper integral

Example

- The improper integral $\int_0^\infty \sin x dx$ does not converge. Indeed, it holds that $\int_0^\beta \sin x dx = [-\cos x]_0^\beta = -\cos\beta + 1$, and as $\beta \to \infty$, $-\cos\beta$ oscillates and does not converge to any value.
- Consider $\int_0^\infty e^{-x} dx$.

Improper integral

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- Consider $\int_0^\infty e^{-x} dx$. For $\beta > 0$, we have $\int_0^\beta e^{-x} dx = [-e^{-x}]_0^\beta = -e^{-\beta} - (-1) = 1 - e^{-\beta}$, hence as $\beta \to \infty$, this tends to 1. That is, $\int_0^\infty e^{-x} dx = 1$.

Improper integral

Example

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- Consider $(-\infty, \infty)$.

$$\int_\alpha^\beta xe^{-x^2} dx = -\frac{1}{2}[e^{-x^2}]_\alpha^\beta = -\frac{1}{2}(e^{-\beta^2} - e^{-\alpha^2}).$$

and both limits $\lim_{\alpha\to-\infty}, \lim_{\beta\to\infty}$ exist. Furthermore, $\int_{-\infty}^\infty xe^{-x^2} dx = -\frac{1}{2}([e^{-x^2}]_{-\infty}^0 + [e^{-x^2}]_0^\infty) = 0$.

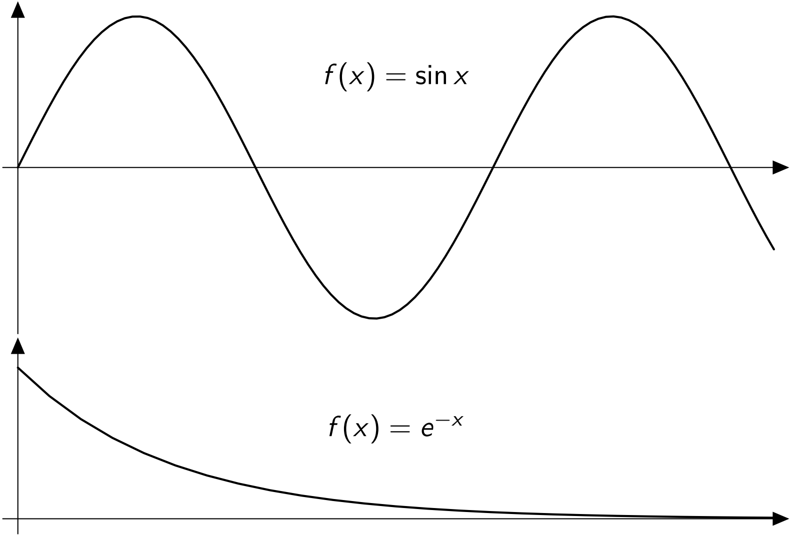

Figure: Convergent and non convergent improper integrals.

Improper integral

Theorem

Let f, g be integrable on all $[\alpha, \beta] \subset (a, b)$.

1. *Let $0 \leq f \leq g$. If $\int_a^b g(x)dx$ converges, then so does $\int_a^b f(x)dx$. If $\int_a^b f(x)dx$ diverges, then so does $\int_a^b g(x)dx$.*
2. *If $f \geq 0$ and there is $M > 0$ such that $\int_\alpha^\beta f(x)dx < M$ for all $a < \alpha < \beta < b$, then $\int_a^b f(x)dx$ converges.*
3. *Let $0 < f$, and $\lim_{x \to b-} \frac{f(x)}{g(x)} = c \neq 0$. Then $\int_\alpha^b f(x)dx$ exists if and only if $\int_\alpha^b g(x)dx$ exists.*
4. *Let $f > 0$ be decreasing on $[\alpha, \infty)$. $\int_\alpha^\infty f(x)dx$ converges if and only if $\sum_{n=N}^\infty f(n)$ converges for some N.*
5. $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$.

Proof.

1. (i) On any interval $[\alpha, \beta]$ it holds that $\int_\alpha^\beta f(x)dx \le \int_\alpha^\beta g(x)dx$, hence $\int_\alpha^\beta f(x)dx$ is bounded and increases as α, β tend to a, b.
2. (ii) As $f(x) \ge 0$, when $\alpha \to a$ (and $\beta \to b$), the integral $\int_\alpha^\beta f(x)dx$ increases. But as it is bounded by M, it must converge to a certain number $\int_a^b f(x)dx \le M$.
3. (iii) Let $c > 0$ (the other case is analogous). For x_0 close enough to b, it holds that $\frac{c}{2}g(x) \le f \le 2cg(x)$. Hence the claim follows from (i).
4. (iv) We have $f(n+1) \le \int_n^{n+1} f(x)dx \le f(n)$, therefore, $\sum_{n=N}^M \le \int_N^{M+1} f(x)dx \le \sum_{n=N}^{M+1} f(n)$.
5. (v) This follows from $-|f| \le f \le |f|$ and (i) for an interval $[\alpha, \beta]$, then by taking the limits.

□

Improper integral

Now we can show that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent using integral. Indeed, $\int_1^N \frac{1}{x} dx \le \sum_{n=1}^{N-1} \frac{1}{n}$, but $\int_1^N \frac{1}{x} dx = [\log x]_1^N = \log N - 0 \to \infty$, therefore, also $\sum_{n=1}^{N-1} \frac{1}{n} \to \infty$ as $N \to \infty$.

When the improper integral $\int_a^b |f(x)| dx$, then we say that the improper integral $\int_a^b f(x) dx$ **converges absolutely**.

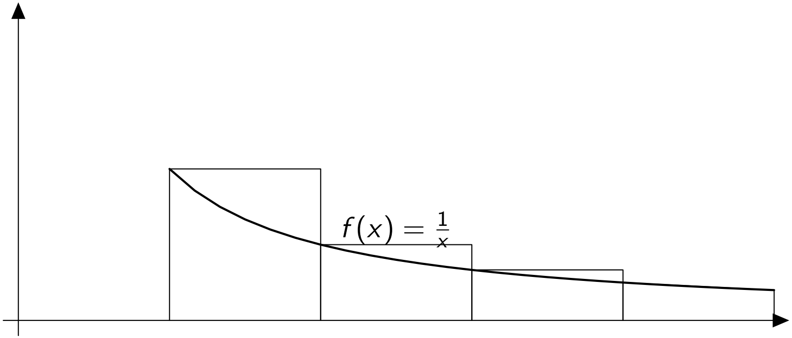

Figure: A graphical proof that $\sum_{n=1}^{N} \frac{1}{n}$ diverges as $N \to \infty$.

Improper integral

Example

- $\int_1^\infty \frac{\cos x}{x^2} dx$

Improper integral

Example

- $\int_1^\infty \frac{\cos x}{x^2} dx$ converges. Indeed, $|\frac{\cos x}{x^2}| \leq \frac{1}{x^2}$ and $\int_1^\alpha \frac{1}{x^2} dx = [-\frac{1}{x}]_1^\alpha = 1 - \frac{1}{\alpha}$ which tends to 1 as $\alpha \to \infty$.
- $\int_1^\infty \frac{\sin x}{x} dx$

Improper integral

Example

- $\int_1^\infty \frac{\cos x}{x^2} dx$ converges. Indeed, $|\frac{\cos x}{x^2}| \leq \frac{1}{x^2}$ and $\int_1^\alpha \frac{1}{x^2} dx = [-\frac{1}{x}]_1^\alpha = 1 - \frac{1}{\alpha}$ which tends to 1 as $\alpha \to \infty$.
- $\int_1^\infty \frac{\sin x}{x} dx$ converges. Indeed, by integration by parts,

$$\int_1^\alpha \frac{\sin x}{x} dx = \left[\frac{-\cos x}{x}\right]_1^\alpha - \int_1^\alpha \frac{\cos x}{x^2} dx = \cos 1 - \frac{\cos \alpha}{\alpha} - \int_1^\alpha \frac{\cos}{x^2} dx$$

The first two terms tend to $\cos 1$ while the last one is convergent.

Improper integral

Example

- $\int_1^\infty \left|\frac{\sin x}{x}\right| dx$

Improper integral

Example

- $\int_1^\infty |\frac{\sin x}{x}| dx$ diverges Indeed, $\int_{n\pi}^{(n+1)\pi} |\frac{\sin x}{x}| \geq \frac{1}{(n+1)\pi} \int_0^\pi \sin x dx = \frac{2}{(n+1)\pi}$ and hence we have $\int_1^\alpha |\frac{\sin x}{x}| dx \geq \sum_{n=2}^{[\alpha]} \frac{2}{\pi(n+1)} \to \infty$.
- $\int_0^\infty \frac{1}{\sqrt{x^3+1}} dx$

Improper integral

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- $\int_0^\infty \frac{1}{\sqrt{x^3+1}}dx$ is convergent. Indeed, it is enough to consider $\int_1^\infty \frac{1}{\sqrt{x^3+1}}dx$, and since $\frac{1}{\sqrt{x^3+1}} \leq \frac{1}{\sqrt{x^3}} = \frac{1}{x^{\frac{3}{2}}}$, we have $\int_1^\beta \frac{1}{\sqrt{x^3+1}}dx \leq \int_1^\beta x^{-\frac{3}{2}}dx$, where the right-hand side is convergent.
- $\int_{-\infty}^\infty \frac{1}{\sqrt{x^4+1}}dx$

Improper integral

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- $\int_{-\infty}^\infty \frac{1}{\sqrt{x^4+1}} dx$ is convergent. Indeed, $\frac{1}{\sqrt{x^4+1}} \leq \frac{1}{\sqrt{x^4}} = \frac{1}{x^2}$ and $\int_1^\beta \frac{1}{x^2} dx = \int_{-\beta}^{-1} \frac{1}{x^2} dx = [-x^{-1}]_1^\beta = 1 - \frac{1}{\beta} \to 1$ as $\beta \to \infty$.

Improper integral

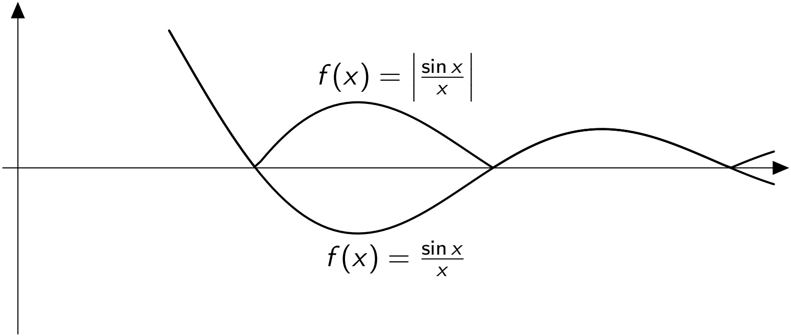

Figure: $f(x) = \frac{\sin x}{x}$ and $f(x) = \left|\frac{\sin x}{x}\right|$. The improper integral of the former in $[1, \infty)$ is convergent, while the latter is not.

Area of a region

We know the are of rectangles, triangles and disks Let us define the area of a more general region.

Definition

- Let $f \geq g$ be two integrable functions on an interval I. The area of the region between g, f is defined by the following:

$$D_{g,f} := \{(x, y) \in \mathbb{R}^2 : x \in I, g(x) \leq y \leq f(x)\}$$

$$\mathrm{area}(D_{g,f}) := \int_I (f(x) - g(x))dx.$$

- Even if I is not bounded, if the improper integral $\int_I (f(x) - g(x))dx$ exists, then we define the area of the region $D_{g,f} = \{(x, y) \in \mathbb{R}^2 : x \in I, g(x) \leq y \leq f(x)\}$ by the same formula.
- If D is the union of such regions, then $\mathrm{area}(D)$ is the sum of the areas of such regions.

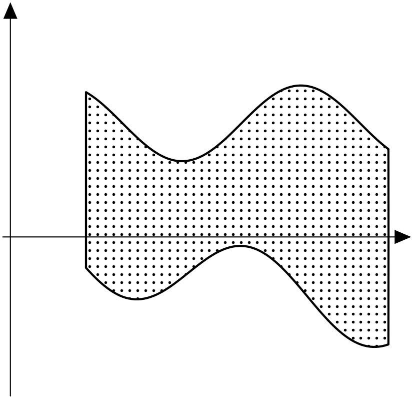

Figure: The area of the region between two functions.

Exercises

- Calculate the following improper integral. $\int_0^\infty x^{\frac{1}{3}} e^{x^{\frac{4}{3}}} dx$
- Calculate the following improper integral. $\int_1^\infty \frac{\log x}{x^2} dx$
- Determine whether the following improper integral converges. $\int_1^\infty \frac{x^3}{x^4+1} dx$.
- Determine whether the following improper integral converges. $\int_1^2 \frac{x^2}{(x-1)^{\frac{1}{2}}} dx$.
- Determine whether the following improper integral converges. $\int_0^1 \frac{x^2}{\sin x - x} dx$.