Mathematical Analysis I: Lecture 40

Lecturer: Yoh Tanimoto

27/11/2020

Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 10:00–11:30.
- Office hour: Tuesday 11:30–12:30.

- Basic Mathematics: first few lessons on
  - Tuesday (14:00 – 16:00 CET): Inequalities, Limits and Derivatives
  - Wednesday (14:00 – 16:00 CET): Study of function

  and then upon request.

Exercises

Calculate the indefinite integral. $\int xe^x dx$.

Solution.

Calculate the indefinite integral. $\int xe^x dx$.

Solution. By integration by parts,

$$\begin{aligned}\int xe^x dx &= xe^x - \int e^x dx + C \\ &= xe^x - e^x + C.\end{aligned}$$

Calculate the indefinite integral. $\int e^x \sin x dx$.
Solution.

Calculate the indefinite integral. $\int e^x \sin x dx$.

Solution. By integration by parts,

$$\begin{aligned}\int e^x \sin x dx &= e^x \sin x - \int e^x \cos x dx + C \\ &= e^x \sin x - \left(e^x \cos x - \int e^x(-\sin x) dx\right) + C,\end{aligned}$$

hence $\int e^x \sin x dx = \frac{1}{2}(e^x(\sin x - \cos x)) + C$.

Calculate the definite integral. $\int_0^1 x^2 e^{-x} dx$.

Solution.

Calculate the definite integral. $\int_0^1 x^2 e^{-x} dx$.
Solution. By integration by parts,

$$\begin{aligned}\int_0^1 x^2 e^{-x} dx &= [-x^2 e^{-x}]_0^1 + \int_0^1 2xe^{-x} dx \\ &= -\frac{1}{e} - [2xe^{-x}]_0^1 + \int_0^1 2e^{-x} dx \\ &= -\frac{3}{e} - [2e^{-x}]_0^1 = 2 - \frac{5}{e}\end{aligned}$$

Calculate the indefinite integral. $\int x\sqrt{1-x^2}dx$.
Solution.

Calculate the indefinite integral. $\int x\sqrt{1-x^2}dx$.
Solution. By substitution $\varphi(x) = -x^2, \varphi'(x) = -2x$,

$$\begin{aligned}\int x\sqrt{1-x^2}dx &= -\frac{1}{2}\int(-2x)\sqrt{1-x^2}dx = -\frac{1}{2}\cdot\frac{2}{3}(1-x^2)^{\frac{3}{2}} + C \\ &= -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + C.\end{aligned}$$

Calculate the indefinite integral. $\int xe^{x^2} dx$.
Solution.

Calculate the indefinite integral. $\int xe^{x^2}dx$.
Solution. By substitution $\varphi(x) = x^2, \varphi'(x) = 2x$,

$$\int xe^{x^2}dx = \frac{1}{2}\int 2xe^{x^2}dx = \frac{1}{2}e^{x^2} + C$$

Calculate the definite integral. $\int_0^1 x^3 e^{x^2} dx$.

Solution.

Calculate the definite integral. $\int_0^1 x^3 e^{x^2}\,dx$.
Solution. By integration by parts and substitution $\varphi(x) = x^2, \varphi'(x) = 2x$,

$$\int_0^1 x^3 e^{x^2}\,dx = \frac{1}{2}\int_0^1 x^2 \cdot 2xe^{x^2}\,dx = \frac{1}{2}[x^2 e^{x^2}]_0^1 - \frac{1}{2}\int_0^1 2xe^{x^2}\,dx$$
$$= \frac{e}{2} - \frac{1}{2}[e^{x^2}]_0^1 = \frac{1}{2}.$$

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$$\frac{x}{x^2-1} = \frac{x}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)},$$

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and from this we have
$x = A(x+1) + B(x-1) = (A+B)x + A - B$,

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$x = A(x+1) + B(x-1) = (A+B)x + A - B$,therefore,

$A - B = 0, A + B = 1$, and $A = \frac{1}{2}, B = \frac{1}{2}$.

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$$\int \frac{x}{x^2-1} dx = \int \left(\frac{1}{2(x-1)} + \frac{1}{2(x+1)} \right) dx$$
$$= \frac{1}{2}(\log|x-1| + \log|x+1|) + C.$$

Calculate the definite integral. $\int_0^1 \frac{1}{x^3-2x^2+x-2}dx$.
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Solution. Let us find the partial fractions.

Calculate the definite integral. $\int_0^1 \frac{1}{x^3-2x^2+x-2}dx$.
Solution. Let us find the partial fractions. $x^3 - 2x^2 + x - 2 = (x^2 + 1)(x - 2)$.

Calculate the definite integral. $\int_0^1 \frac{1}{x^3-2x^2+x-2}dx$.
Solution. Let us find the partial fractions.$x^3 - 2x^2 + x - 2 = (x^2 + 1)(x - 2)$.

$$\frac{1}{x^3 - 2x^2 + x - 2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 2} = \frac{(Ax + B)(x - 2) + C(x^2 + 1)}{(x^2 + 1)(x - 2)}$$

and from this we have $1 = (A + C)x^2 + (B - 2A)x + (C - 2B)$,

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Calculate the definite integral. $\int_0^1 \frac{1}{x^3-2x^2+x-2}dx$.
Solution. Let us find the partial fractions. $x^3 - 2x^2 + x - 2 = (x^2+1)(x-2)$.

$$\frac{1}{x^3-2x^2+x-2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-2} = \frac{(Ax+B)(x-2)+C(x^2+1)}{(x^2+1)(x-2)}$$

and from this we have $1 = (A+C)x^2 + (B-2A)x + (C-2B)$,and hence $A+C=0, B-2A=0, C-2B=1$. By solving this, $C = \frac{1}{5}, A = -\frac{1}{5}, B = -\frac{2}{5}$. That is,

$$\int \frac{1}{x^3-2x^2+x-2}dx = \frac{1}{5}\int \left(\frac{-x-2}{x^2+1} + \frac{1}{x-2}\right) dx$$
$$= \frac{1}{10}(-\log(x^2+1) - 4\arctan x + 2\log|x-2|).$$

Therefore, $\int_0^1 \frac{1}{x^3-2x^2+x-2}dx = -\frac{\pi}{10} - \frac{3\log 2}{10}$.

Calculate the indefinite integral. $\int \frac{1}{\cos x} dx$.

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$$\begin{aligned}\int \frac{1}{\cos x} dx &= \int \frac{1}{\cos^2 x} \cos x dx \\ &= \int \frac{1}{1 - \sin^2 x} \varphi'(x) dx = \int \frac{1}{1 - t^2} dt \\ &= \frac{1}{2} \int (\frac{1}{1+t} + \frac{1}{1-t}) dt = \frac{1}{2} \log \frac{|1+t|}{|1-t|}.\end{aligned}$$

That is, $\int \frac{1}{\cos x} dx = \frac{1}{2} \log \frac{|\sin x + 1|}{|\sin x - 1|}$.

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That is, $\int \frac{1}{\cos x} dx = \frac{1}{2} \log \frac{|\sin x + 1|}{|\sin x - 1|}$.

Solution 2. By change of variables $x = \varphi(t) = 2 \arctan t$, or $t = \tan \frac{x}{2}$, $\varphi'(t) = \frac{2}{t^2+1}, \cos x = \frac{1-t^2}{t^2+1}$,

$$\int \frac{1}{\cos x} dx = \int \frac{2}{1-t^2} dt = \int (\frac{1}{t+1} - \frac{1}{t-1}) dt = \log \frac{|t+1|}{|t-1|}.$$

That is, $\int \frac{1}{\cos x} dx = \log \frac{|\tan \frac{x}{2} + 1|}{|\tan \frac{x}{2} - 1|}$.

Calculate the indefinite integral. $\int \frac{1}{\cos x + \sin x} dx$.
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Solution. By change of variables

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Solution. By change of variables
$t = \varphi(x) = \tan\frac{x}{2}, \varphi'(x) = \frac{2}{t^2+1}, \cos x = \frac{1-t^2}{t^2+1}, \sin x = \frac{2t}{t^2+1}$,

$$\int \frac{1}{\cos x + \sin x}dx = \int \frac{1}{\frac{1-t^2}{t^2+1} + \frac{2t}{t^2+1}} \cdot \frac{2}{t^2+1}dt$$
$$= -\int \frac{2}{t^2-2t-1}dt$$
$$= -\frac{1}{\sqrt{2}}\int \left(\frac{1}{t-1-\sqrt{2}} - \frac{1}{t-1+\sqrt{2}}\right)dt$$
$$= -\frac{1}{\sqrt{2}}\log\frac{|t-1-\sqrt{2}|}{|t-1+\sqrt{2}|}.$$

Calculate the indefinite integral. $\int \frac{1}{\cos x + \sin x} dx$.
Solution. By change of variables
$t = \varphi(x) = \tan \frac{x}{2}, \varphi'(x) = \frac{2}{t^2+1}, \cos x = \frac{1-t^2}{t^2+1}, \sin x = \frac{2t}{t^2+1}$,

$$\begin{aligned}\int \frac{1}{\cos x + \sin x} dx &= \int \frac{1}{\frac{1-t^2}{t^2+1} + \frac{2t}{t^2+1}} \cdot \frac{2}{t^2+1} dt \\ &= -\int \frac{2}{t^2 - 2t - 1} dt \\ &= -\frac{1}{\sqrt{2}} \int \left(\frac{1}{t-1-\sqrt{2}} - \frac{1}{t-1+\sqrt{2}} \right) dt \\ &= -\frac{1}{\sqrt{2}} \log \frac{|t-1-\sqrt{2}|}{|t-1+\sqrt{2}|}.\end{aligned}$$

That is, $\int \frac{1}{\cos x + \sin x} dx = -\frac{1}{\sqrt{2}} \log \frac{|\tan \frac{x}{2} - 1 - \sqrt{2}|}{|\tan \frac{x}{2} - 1 + \sqrt{2}|}$.

Calculate the definite integral. $\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{4 - x^2} dx$.
Solution.

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Solution. Change of variables $x = 2\sin t$.

Calculate the definite integral. $\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{4 - x^2} dx$.
Solution. Change of variables $x = 2 \sin t$. Note that $\sin(-\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}, \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Therefore, with $\frac{dx}{dt} = 2 \cos t$,

Calculate the definite integral. $\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{4-x^2}dx$.
Solution. Change of variables $x = 2\sin t$. Note that $\sin(-\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}, \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Therefore, with $\frac{dx}{dt} = 2\cos t$,

$$\begin{aligned}\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{4-x^2}dx &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{4-4\sin^2 x} \cdot 2\cos t dt \\ &= 4\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 t dt \\ &= 4\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos(2t)+1}{2} dt \\ &= 4\left[\frac{\sin(2t)}{4} + \frac{t}{2}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 2+\pi\end{aligned}$$

Calculate the indefinite integral. $\int_0^2 \sqrt{8 - x^2} dx$.

Solution. Solution. Change of variables $x = 2\sqrt{2} \sin t$. Note that $\sin 0 = 0, \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Therefore, with $\frac{dx}{dt} = 2\sqrt{2} \cos t$,

$$\begin{aligned} \int_0^2 \sqrt{4 - x^2} dx &= \int_0^{\frac{\pi}{4}} \sqrt{8 - 8 \sin^2 x} \cdot 2\sqrt{2} \cos t dt \\ &= 8 \int_0^{\frac{\pi}{4}} \cos^2 t dt \\ &= 8 \int_0^{\frac{\pi}{4}} \frac{\cos(2t) + 1}{2} dt \\ &= 8 \left[\frac{\sin(2t)}{4} + \frac{t}{2} \right]_0^{\frac{\pi}{4}} = 2 + \pi \end{aligned}$$

Calculate the integral. $\int_{-1}^{1} \sin(\sin x) dx$.
Solution.

Calculate the integral. $\int_{-1}^{1} \sin(\sin x) dx$.
Solution. Note that $\sin(\sin(-x)) = \sin(-\sin(x)) = -\sin(\sin x)$, and the interval $[-1, 1]$ is symmetric, hence this is 0.

Calculate the inproper integral. $\int_0^\infty xe^{-x}dx$.
Solution.

Calculate the inproper integral. $\int_0^\infty xe^{-x}dx$.
Solution. Note that with $F(x) = -e^{-x} - xe^{-x}$, $F'(x) = xe^{-x}$. Therefore,

$$\int_0^\beta xe^{-x}dx = [-e^{-x} - xe^{-x}]_0^\beta = -e^{-\beta} - \beta e^{-\beta} + 1$$

and as $\beta \to \infty$, this tends to 1. Hence $\int_0^\infty xe^{-x}dx = 1$.