Mathematical Analysis I: Lecture 39

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Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 10:00–11:30.
- Office hour: Tuesday 11:30–12:30.

- Basic Mathematics: first few lessons on
  - Tuesday (14:00 – 16:00 CET): Inequalities, Limits and Derivatives
  - Wednesday (14:00 – 16:00 CET): Study of function

  and then upon request.

- Today: Apostol Vol. 1, Chapter 5, 6.8, 7.5,

Change of variables

When the function contains $\sin x$ and $\cos x$, it is often useful to do the change of variable $x = \varphi(t) = 2 \arctan t$, or $t = \tan \frac{x}{2}$. Indeed, we have $\varphi'(t) = \frac{2}{1+t^2}$, while $\frac{1}{\cos^2 \frac{x}{2}} = \frac{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} = 1 + t^2$ and
$\sin x = \sin(2 \cdot \frac{x}{2}) = 2 \sin \frac{x}{2} \cos \frac{x}{2} = \frac{2t}{1+t^2}$ and
$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \frac{1-t^2}{1+t^2}$.
For example,

$$\int \frac{1}{\sin x} dx = \int \frac{t^2+1}{2t} \cdot \frac{2}{1+t^2} dt = \log |t| + C = \log \left| \tan \frac{x}{2} \right| + C.$$

Change of variables

Corollary

Let f be contiuous on $[a,b]$, φ differentiable and φ' continuous on $[\alpha,\beta]$, and $\varphi([\alpha,\beta]) \subset [a,b]$, $\varphi(\alpha) = a, \varphi(\beta) = b$. Then

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t)) \cdot \varphi'(t)dt$$

Proof.

Let $F(x) = \int_a^x f(x)dx$. Since $\frac{d}{dt}(F(\varphi(t))) = f(\varphi(t)) \cdot \varphi'(t)$,

$$\int_\alpha^\beta f(\varphi(t)) \cdot \varphi'(t)dt = [F(\varphi(t))]_\alpha^\beta = [F(x)]_a^b = \int_a^b f(x)dx.$$

□

Change of variables

Example

- Note that $\sqrt{1-\sin^2 t} = |\cos t|$ and this is equal to $\cos t$ if $|t| < \frac{\pi}{2}$, hence with $x = \sin t$,

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}dx &= \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\cos t\,dt = \int_0^{\frac{\pi}{2}} \cos^2 t\,dt \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos(2t)+1}{2}dt = \left[\frac{\sin(2t)}{4} + \frac{t}{2}\right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{4}.\end{aligned}$$

Some remarks

- If $f(x) = f(-x)$, then by the change of variables $x = -t$, $\int_{-a}^{0} f(x)dx = \int_{a}^{0} f(-t)(-t)'dt = \int_{0}^{a} f(t)dt$, hence $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.
For example, $\int_{-1}^{1} \sqrt{1-x^2}dx = 2\int_{0}^{1} \sqrt{1-x^2}dx = \frac{\pi}{2}$.
- If $f(x) = -f(-x)$, then by the change of variables $x = -t$, $\int_{-a}^{0} f(x)dx = \int_{a}^{0} f(-t)(-t)'dt = -\int_{0}^{a} f(t)dt$, hence $\int_{-a}^{a} f(x)dx = 0$.
For example, $\int_{-1}^{1} e^{x^2} \sin x dx = 0$.

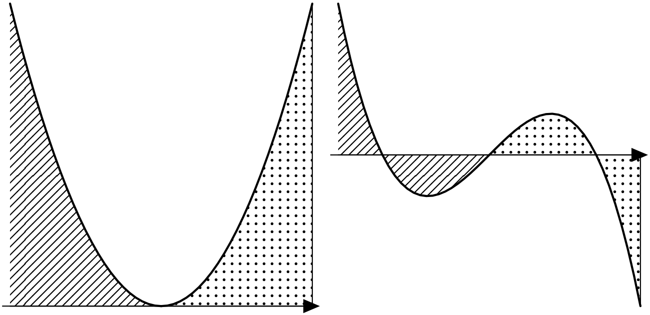

Figure: Integral of symmetric and antisymmetric functions.

Some remarks

- Logarithmic differentiation: If $f(x)$ is difficult to differentiate but $\log f(x)$ is easy, then we have $D(\log f(x)) = \frac{f'(x)}{f(x)}$, hence we have $f'(x) = f(x)D(\log f(x))$.
  For example, $f(x) = x^x$ (for $x > 0$) is not a simple product or a composition. But $\log f(x) = x \log x$, hence $D(\log f(x)) = \log x + 1$, hence $f'(x) = x^x(\log x + 1)$.

Taylor formula with remainder

Theorem

If f is differentiable $n+1$ times in an neighbourhood of x_0, then for x in that neighbourhood,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_n(x, x_0),$$

where $R_n(x, x_0) = \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(y)(x-y)^n dy$.

This is interesting, because for some functions, we can prove that the Taylor series *converges* to the original function.

Proof.

This is true for $n = 0$, because

$$f(x_0) + \int_{x_0}^{x} f'(y)dy = f(x_0) + [f(y)]_{x_0}^{x} = f(x).$$

To prove the formula by induction, assume the claim for n and let f be $n+2$ times differentiable, then $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x,x_0)$,

$$\begin{aligned}
R_n(x,x_0) &= \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(y)(x-y)^n dy \\
&= -\frac{1}{(n+1)!}\left[f^{(n+1)}(y)(x-y)^{n+1}\right]_{x_0}^{x} \\
&\quad + \frac{1}{(n+1)!}\int_{x_0}^{x} f^{(n+2)}(y)(x-y)^{n+1}dy \\
&= \frac{1}{(n+1)!}f^{(n+1)}(x_0)(x-x_0)^{n+1} + \frac{1}{(n+1)!}\int_{x_0}^{x} f^{(n+2)}(y)(x-y)^{n+1}dy
\end{aligned}$$

□

Taylor formula with remainder

Let us take $x_0 = 0$ and consider the interval $(-R, R)$.

- $e^x = \sum_{k=0}^n \frac{x^k}{k!} + \frac{1}{(n+1)!}\int_0^x e^y (y-x)^n dx$. As $|x| < R$, we have $e^y < e^R$ and $|(y-x)^n| < R^n$. Altogether, the remainder term is

$$\frac{1}{(n+1)!}\left|\int_0^x e^y (y-x)^n dx\right| \le \frac{1}{(n+1)!}\left|\int_0^x e^R R^n dx\right| \le \frac{e^R R^{n+1}}{(n+1)!}.$$

Note that for any R, $\frac{R^{n+1}}{(n+1)!} \to 0$ because for sufficiently large n we have $n > 2R$, hence from that point the sequence decreases more than by $\frac{1}{2}$. This means that $e^x - \sum_{k=0}^n \frac{x^k}{k!} \to 0$, that is, the Taylor series converges to e^x for $x \in (-R, R)$, and this is denoted by

$$e^x = \sum_{n=0}^\infty \frac{x^n}{n!}.$$

Furthermore, R was arbitrary, hence this holds for any x.

- The same argument holds for $\sin x, \cos x$, because $|D^n(\sin x)| \le 1, |D^n(\cos x)| \le 1$. That is,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!},$$

  in the sense that for each x the series converges to the original function.

More properties of Taylor *series* will be studied in Mathematical Analysis II.

Taylor formula with remainder

We can also find the approximate value of $e = e^1$ using the Taylor formula with remainder. We know that $e < (1+\frac{1}{n})^{n+1}$ for any n. In particular, $e < 4$. Therefore, by

$$e^1 = \sum_{k=0}^{12} \frac{1^k}{k!} + \frac{1}{13!}\int_0^1 e^y (1-y)^{12} dy$$

and the error term satisfies $0 < \frac{1}{13!}\int_0^1 e^y(1-y)^{12}dy < \frac{4}{13!} < 0.0000002$. Therefore, the approximation of e,

$$\sum_{k=0}^{12} \frac{1^k}{k!} \cong 2.71828182,$$

is correct up to the 7-th digit.

Improper integral

We can define integral for (some) funtions that are not bounded and on an interval not bounded.

Definition

Let (a, b) be an interval, $a \in \mathbb{R}$ or $a = -\infty$ and $b \in \mathbb{R}$ or $b = +\infty$. Let f be a function integrable on all $[\alpha, \beta]$, where $a < \alpha < \beta < b, \alpha, \beta \in \mathbb{R}$. If there exists the limit $\lim_{\alpha \to a} \int_\alpha^\beta f(x)dx$, then we denote it by

$$\int_a^\beta f(x)dx = \lim_{\alpha \to a} \int_\alpha^\beta f(x)dx.$$

It also holds that $\int_a^\gamma f(x)dx = \int_a^\beta f(x)dx + \int_\beta^\gamma f(x)dx$ for $\gamma \in (a, b)$. Analogously if there exists the limit $\lim_{\beta \to b} \int_\alpha^\beta f(x)dx$, then we write $\int_\alpha^b f(x)dx = \lim_{\beta \to b} \int_\alpha^\beta f(x)dx$. If both limits exist, then we denote $\int_a^b f(x)dx = \int_a^{x_0} f(x)dx + \int_{x_0}^b f(x)dx$ for some $x_0 \in (a, b)$.

Improper integral

This definition does not depend on $x_0 \in (a, b)$. Indeed,

$$
\begin{aligned}
&\int_a^{x_0} f(x)dx + \int_{x_0}^b f(x)dx \\
&= \int_a^{x_0} f(x)dx + \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_0} f(x)dx + \int_{x_0}^b f(x)dx \\
&= \int_a^{x_1} f(x)dx + \int_{x_1}^b f(x)dx.
\end{aligned}
$$

Improper integral

Example

- Consider $(0, 1)$ and the function $f(x) = x^\alpha, \alpha \in \mathbb{R}$. For $\varepsilon > 0$, if $\alpha \neq -1$,

$$\int_\varepsilon^1 x^\alpha dx = \frac{1}{\alpha+1}[x^{\alpha+1}]_\varepsilon^1 = \frac{1}{\alpha+1}(1 - \varepsilon^{\alpha+1}),$$

and as $\varepsilon \to +0$, this tends to $\frac{1}{\alpha+1}$ if $\alpha > -1$, and diverges if $\alpha < -1$. If $\alpha = -1$,

$$\int_\varepsilon^1 x^{-1} dx = [\log x]_\varepsilon^1 = -\log \varepsilon,$$

and this tends to ∞ as $\varepsilon \to 0+$. Therefore, for $\alpha > -1, \int_0^1 x^\alpha dx = \frac{1}{\alpha+1}$.

Improper integral

Example

- Consider $(1, \infty)$ and the function $f(x) = x^\alpha, \alpha \in \mathbb{R}$. For $\beta > 1$, if $\alpha \neq -1$,

$$\int_1^\beta x^\alpha dx = \frac{1}{\alpha+1}[x^{\alpha+1}]_1^\beta = \frac{1}{\alpha+1}(\beta^{\alpha+1} - 1),$$

and as $\beta \to +\infty$, this tends to $-\frac{1}{\alpha+1} = \frac{1}{|\alpha+1|}$ if $\alpha < -1$, and diverges if $\alpha > -1$. For $\alpha = -1$,

$$\int_1^\beta x^{-1} dx = [\log x]_1^\beta = \log \beta,$$

and this tends to ∞ as $\beta \to +\infty$. Therefore, $\int_1^\infty f(x)dx = \frac{1}{|\alpha+1|}$ for $\alpha < -1$.

Exercises

- Calculate the integral. $\int_{-1}^{1} \sin(\sin x) dx$.
- Calculate the indefinite integral. $\int_{0}^{2} \sqrt{8 - x^2} dx$.
- Calculate the inproper integral. $\int_{0}^{\infty} x e^{-x} dx$.