Mathematical Analysis I: Lecture 38

Lecturer: Yoh Tanimoto

25/11/2020

Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 10:00–11:30.
- Office hour: Tuesday 11:30–12:30.

- Basic Mathematics: first few lessons on
  - Tuesday (14:00 – 16:00 CET): Inequalities, Limits and Derivatives
  - Wednesday (14:00 – 16:00 CET): Study of function

  and then upon request.

- Today: Apostol Vol. 1, Chapter 5,6.23.

Rational functions

We know that

- $\int \frac{1}{(x-a)} dx = \log |x - a|$.
- for $n \in \mathbb{N}, n \geq 2$, $\int \frac{1}{(x-a)^n} dx = \frac{-1}{(n-1)(x-a)^{n-1}}$.
- $\int \frac{1}{x^2+1} dx = \arctan x$, $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \arctan(\frac{x}{a})$.
- $\int \frac{1}{(x-b)^2+a^2} dx = \frac{1}{a} \arctan(\frac{(x-b)}{a})$.

In general, the derivative of the primitive F of f must be the original function f. We can check that the primitive in this way. This is often useful because the calculus of primitive is often complicated, while derivative is mechanical.

Rational functions

We also have

$$\int \frac{1}{(x^2+1)^2}dx = \int \frac{1+x^2-x^2}{(x^2+1)^2}dx = \int \frac{1}{(x^2+1)} + \int \frac{-2x\cdot x}{2(x^2+1)^2}dx$$
$$= \arctan x + \frac{x}{2(x^2+1)} - \int \frac{1}{2(x^2+1)}$$
$$= \arctan x + \frac{x}{2(x^2+1)} - \frac{1}{2}\arctan x$$
$$= \frac{1}{2}\arctan x + \frac{x}{2(x^2+1)}.$$

Indeed, by taking the derivative,

$$\left(\frac{1}{2}\arctan x + \frac{x}{2(x^2+1)}\right)' = \frac{1}{2(x^2+1)} + \frac{(x^2+1)-2x^2}{2(x^2+1)^2} = \frac{1}{x^2+1}.$$

Rational functions

Example

- $\int \frac{x^3-1}{4x^3-x}dx = \int \frac{\frac{1}{4}(4x^3-x)+\frac{x}{4}-1}{4x^3-x}dx.$

Example

- $\int \frac{x^3-1}{4x^3-x}dx = \int \frac{\frac{1}{4}(4x^3-x)+\frac{x}{4}-1}{4x^3-x}dx$. Note that $\frac{x-4}{4x^3-x} = \frac{x-4}{x(2x-1)(2x+1)} = \frac{4}{x} + \frac{-\frac{7}{2}}{2x-1} + \frac{-\frac{9}{2}}{2x+1}$ (see below) and hence

Example

- $\int \frac{x^3-1}{4x^3-x}dx = \int \frac{\frac{1}{4}(4x^3-x)+\frac{x}{4}-1}{4x^3-x}dx$. Note that $\frac{x-4}{4x^3-x} = \frac{x-4}{x(2x-1)(2x+1)} = \frac{4}{x} + \frac{-\frac{7}{2}}{2x-1} + \frac{-\frac{9}{2}}{2x+1}$ (see below) and hence

$$\int \frac{x^3-1}{4x^3-x}dx = \frac{x}{4} + \frac{1}{4}\int \left(\frac{4}{x} + \frac{-\frac{7}{2}}{2x-1} + \frac{-\frac{9}{2}}{2x+1}\right) dx$$
$$= \frac{x}{4} + \log|x| - \frac{7}{16}\log|2x-1| - \frac{9}{16}\log|2x+1|.$$

Rational functions

Example

- Using $\frac{1}{(x-1)^2(x^2+1)} = \frac{Ax+B}{(x-1)^2} + \frac{Cx+D}{x^2+1} = \frac{-\frac{x}{2}+1}{(x-1)^2} + \frac{\frac{x}{2}}{x^2+1}$ we get

Rational functions

Example

- Using $\frac{1}{(x-1)^2(x^2+1)} = \frac{Ax+B}{(x-1)^2} + \frac{Cx+D}{x^2+1} = \frac{-\frac{x}{2}+1}{(x-1)^2} + \frac{\frac{x}{2}}{x^2+1}$ we get

$$\begin{aligned}\int \frac{1}{(x-1)^2(x^2+1)}dx &= \int \frac{-\frac{x}{2}+1}{(x-1)^2}dx + \int \frac{\frac{x}{2}}{x^2+1}dx \\ &= \int \frac{-\frac{(x-1)}{2}+\frac{1}{2}}{(x-1)^2}dx + \int \frac{\frac{x}{2}}{x^2+1}dx \\ &= -\frac{1}{2}\log|x-1| - \frac{1}{2(x-1)} + \frac{1}{4}\log(x^2+1).\end{aligned}$$

Rational functions

In general, if $P(x)$ and $Q(x)$ are polynomials, $\frac{P(x)}{Q(x)}$ can be written as a sum of $\frac{P_1(x)}{(x-a)^n}$ or $\frac{P_2(x)}{((x-b)^2+a^2)^n}$ (with different polynomials P_1, P_2), and for each of them one can find a primitive.

Example

- $\frac{1}{(x-1)(x+1)}$.

Rational functions

In general, if $P(x)$ and $Q(x)$ are polynomials, $\frac{P(x)}{Q(x)}$ can be written as a sum of $\frac{P_1(x)}{(x-a)^n}$ or $\frac{P_2(x)}{((x-b)^2+a^2)^n}$ (with different polynomials P_1, P_2), and for each of them one can find a primitive.

Example

- $\frac{1}{(x-1)(x+1)}$. We put $\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$.

Rational functions

In general, if $P(x)$ and $Q(x)$ are polynomials, $\frac{P(x)}{Q(x)}$ can be written as a sum of $\frac{P_1(x)}{(x-a)^n}$ or $\frac{P_2(x)}{((x-b)^2+a^2)^n}$ (with different polynomials P_1, P_2), and for each of them one can find a primitive.

Example

- $\frac{1}{(x-1)(x+1)}$. We put $\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$. Then

$$\frac{1}{(x-1)(x+1)} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$$

Rational functions

In general, if $P(x)$ and $Q(x)$ are polynomials, $\frac{P(x)}{Q(x)}$ can be written as a sum of $\frac{P_1(x)}{(x-a)^n}$ or $\frac{P_2(x)}{((x-b)^2+a^2)^n}$ (with different polynomials P_1, P_2), and for each of them one can find a primitive.

Example

- $\frac{1}{(x-1)(x+1)}$. We put $\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$. Then

$$\frac{1}{(x-1)(x+1)} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$$

and $1 = (A+B)x + (A-B)$. This means that

Rational functions

In general, if $P(x)$ and $Q(x)$ are polynomials, $\frac{P(x)}{Q(x)}$ can be written as a sum of $\frac{P_1(x)}{(x-a)^n}$ or $\frac{P_2(x)}{((x-b)^2+a^2)^n}$ (with different polynomials P_1, P_2), and for each of them one can find a primitive.

Example

- $\frac{1}{(x-1)(x+1)}$. We put $\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$. Then

$$\frac{1}{(x-1)(x+1)} = \frac{A(x+1) + B(x-1)}{(x-1)(x+1)}$$

and $1 = (A+B)x + (A-B)$. This means that $A+B = 0$ and $1 = A - B$,

Rational functions

In general, if $P(x)$ and $Q(x)$ are polynomials, $\frac{P(x)}{Q(x)}$ can be written as a sum of $\frac{P_1(x)}{(x-a)^n}$ or $\frac{P_2(x)}{((x-b)^2+a^2)^n}$ (with different polynomials P_1, P_2), and for each of them one can find a primitive.

Example

- $\frac{1}{(x-1)(x+1)}$. We put $\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$. Then

$$\frac{1}{(x-1)(x+1)} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$$

and $1 = (A+B)x + (A-B)$. This means that $A+B=0$ and $1 = A-B$, therfore, $A = \frac{1}{2}, B = -\frac{1}{2}$.

Rational functions

Example

- $\frac{1}{(x+1)(x^2+1)^2}$.

Rational functions

Example

- $\frac{1}{(x+1)(x^2+1)^2}$. Put $\frac{1}{(x+1)(x^2+1)^2} = \frac{A}{x+1} + \frac{Bx^3+Cx^2+Dx+E}{(x^2+1)^2}$

Rational functions

Example

- $\frac{1}{(x+1)(x^2+1)^2}$. Put $\frac{1}{(x+1)(x^2+1)^2} = \frac{A}{x+1} + \frac{Bx^3+Cx^2+Dx+E}{(x^2+1)^2}$ and

$$\frac{1}{(x+1)(x^2+1)^2} = \frac{A(x^2+1)^2 + (Bx^3+Cx^2+Dx+E)(x+1)}{(x+1)(x^2+1)^2}.$$

Rational functions

Example

- $\frac{1}{(x+1)(x^2+1)^2}$. Put $\frac{1}{(x+1)(x^2+1)^2} = \frac{A}{x+1} + \frac{Bx^3+Cx^2+Dx+E}{(x^2+1)^2}$ and

$$\frac{1}{(x+1)(x^2+1)^2} = \frac{A(x^2+1)^2 + (Bx^3+Cx^2+Dx+E)(x+1)}{(x+1)(x^2+1)^2}.$$

$$1 = A(x^4+2x^2+1)+(Bx^4+(B+C)x^3+(C+D)x^2+(D+E)x+E),$$

Rational functions

Example

- $\frac{1}{(x+1)(x^2+1)^2}$. Put $\frac{1}{(x+1)(x^2+1)^2} = \frac{A}{x+1} + \frac{Bx^3+Cx^2+Dx+E}{(x^2+1)^2}$ and

$$\frac{1}{(x+1)(x^2+1)^2} = \frac{A(x^2+1)^2 + (Bx^3+Cx^2+Dx+E)(x+1)}{(x+1)(x^2+1)^2}.$$

$1 = A(x^4+2x^2+1) + (Bx^4 + (B+C)x^3 + (C+D)x^2 + (D+E)x + E)$,
so $A+B=0, B+C=0, 2A+C+D=0, D+E=0, A+E=1$.

Rational functions

Example

- $\frac{1}{(x+1)(x^2+1)^2}$. Put $\frac{1}{(x+1)(x^2+1)^2} = \frac{A}{x+1} + \frac{Bx^3+Cx^2+Dx+E}{(x^2+1)^2}$ and

$$\frac{1}{(x+1)(x^2+1)^2} = \frac{A(x^2+1)^2 + (Bx^3+Cx^2+Dx+E)(x+1)}{(x+1)(x^2+1)^2}.$$

$1 = A(x^4+2x^2+1) + (Bx^4+(B+C)x^3+(C+D)x^2+(D+E)x+E)$, so $A+B=0, B+C=0, 2A+C+D=0, D+E=0, A+E=1$. Observe $A=-B,\ C=-B, D=-E$ and $-3B-E=0, -B+E=1$, so

Example

- $\frac{1}{(x+1)(x^2+1)^2}$. Put $\frac{1}{(x+1)(x^2+1)^2} = \frac{A}{x+1} + \frac{Bx^3+Cx^2+Dx+E}{(x^2+1)^2}$ and

$$\frac{1}{(x+1)(x^2+1)^2} = \frac{A(x^2+1)^2 + (Bx^3+Cx^2+Dx+E)(x+1)}{(x+1)(x^2+1)^2}.$$

$1 = A(x^4+2x^2+1) + (Bx^4 + (B+C)x^3 + (C+D)x^2 + (D+E)x + E)$, so $A+B=0, B+C=0, 2A+C+D=0, D+E=0, A+E=1$. Observe $A=-B$, $C=-B, D=-E$ and $-3B-E=0, -B+E=1$, so $B=-\frac{1}{4}, E=\frac{3}{4}, A=\frac{1}{4}, C=\frac{1}{4}, D=-\frac{3}{4}$.

Rational functions

Example

- $\frac{1}{(x+1)(x^2+1)^2}$. Put $\frac{1}{(x+1)(x^2+1)^2} = \frac{A}{x+1} + \frac{Bx^3+Cx^2+Dx+E}{(x^2+1)^2}$ and

$$\frac{1}{(x+1)(x^2+1)^2} = \frac{A(x^2+1)^2 + (Bx^3+Cx^2+Dx+E)(x+1)}{(x+1)(x^2+1)^2}.$$

$1 = A(x^4+2x^2+1) + (Bx^4 + (B+C)x^3 + (C+D)x^2 + (D+E)x + E)$, so $A+B=0, B+C=0, 2A+C+D=0, D+E=0, A+E=1$. Observe $A=-B$, $C=-B, D=-E$ and $-3B-E=0, -B+E=1$, so $B=-\frac{1}{4}, E=\frac{3}{4}, A=\frac{1}{4}, C=\frac{1}{4}, D=-\frac{3}{4}$. Altogether,

$$\frac{1}{(x+1)(x^2+1)^2} = \frac{\frac{1}{4}}{x+1} + \frac{-\frac{1}{4}x^3 + \frac{1}{4}x^2 - \frac{3}{4}x + \frac{3}{4}}{(x^2+1)^2}$$
$$= \frac{\frac{1}{4}}{x+1} + \frac{-\frac{x}{4}+\frac{1}{4}}{x^2+1} + \frac{-\frac{1}{2}x+\frac{1}{2}}{(x^2+1)^2}$$

Change of variables

Let $F(x)$ be a primitive of $f(x)$, that is $\int f(x)dx = F(x)$. If it is difficult to find F directly, one may consider a change of variables $x = \varphi(t)$. By the chain rule, $\frac{d}{dt}F(\varphi(t)) = f(\varphi(t))\varphi'(t)$. If $G(t)$ is a primitive of $f(\varphi(t))\varphi'(t)$, then $F(x) = G(\varphi^{-1}(t))$.
In order to recall the rule, it is useful to write

$$\int f(x)dx = \int f(\varphi(t))\frac{dx}{dt}dt,$$

even if this is only formal.

Change of variables

Example

- $f(x) = \frac{1}{e^x+1}$. With $t = e^x + 1, x = \varphi(t) = \log(t-1), \varphi'(t) = \frac{1}{t-1}$,

$$\int \frac{1}{e^x+1} dx = \int \frac{1}{t(t-1)} dt = \int \left(\frac{1}{t-1} - \frac{1}{t}\right) dt = \log\left|\frac{t-1}{t}\right|$$

and with $t = e^x + 1$, $\int \frac{1}{e^x+1} dx = \log \frac{e^x}{e^x+1}$.

Change of variables

Example

- $f(x) = \frac{x^2}{\sqrt{1-x^2}}$.

Change of variables

Example

- $f(x) = \frac{x^2}{\sqrt{1-x^2}}$. With $x = \varphi(t) = \sin t, \varphi'(t) = \cos t, t = \arcsin x$, if $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

Change of variables

Example

- $f(x) = \frac{x^2}{\sqrt{1-x^2}}$. With $x = \varphi(t) = \sin t, \varphi'(t) = \cos t, t = \arcsin x$, if $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$$\int \frac{x^2}{\sqrt{1-x^2}} dx = \int \frac{\sin^2 t}{\cos t} \cos t dt$$
$$= \int \sin^2 t dt = \int \frac{1-\cos 2t}{2} dt$$
$$= \frac{t}{2} - \frac{\sin 2t}{4}$$

Change of variables

Example

- $f(x) = \frac{x^2}{\sqrt{1-x^2}}$. With $x = \varphi(t) = \sin t, \varphi'(t) = \cos t, t = \arcsin x$, if $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$$\begin{aligned}\int \frac{x^2}{\sqrt{1-x^2}}dx &= \int \frac{\sin^2 t}{\cos t}\cos t dt \\ &= \int \sin^2 t dt = \int \frac{1-\cos 2t}{2}dt \\ &= \frac{t}{2} - \frac{\sin 2t}{4}\end{aligned}$$

and with $t = \arcsin x, \sin 2t = 2\sin t \cos t = 2x\sqrt{1-x^2}$, we obtain $\int \frac{x^2}{\sqrt{1-x^2}}dx = \frac{1}{2}\arcsin x - \frac{1}{2}x\sqrt{1-x^2}$.

Change of variables (this and the rest will be explained on 26 Nov.)

When the function contains $\sin x$ and $\cos x$, it is often useful to do the change of variable $x = \varphi(t) = 2\arctan t$, or $t = \tan\frac{x}{2}$. Indeed, we have $\varphi'(t) = \frac{2}{1+t^2}$, while $\frac{1}{\cos^2\frac{x}{2}} = \frac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} = 1+t^2$ and
$\sin x = \sin(2\cdot\frac{x}{2}) = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2t}{1+t^2}$ and
$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-t^2}{1+t^2}$.
For example,

$$\int \frac{1}{\sin x}dx = \int \frac{t^2+1}{2t}\cdot\frac{2}{1+t^2}dt = \log t = \log\tan\frac{x}{2}.$$

Change of variables

Corollary

Let f be contiunous on $[a,b]$, φ differentiable and φ' continuous on $[\alpha,\beta]$, and $\varphi([\alpha,\beta]) \subset [a,b]$, $\varphi(\alpha) = a, \varphi(\beta) = b$. Then

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t)) \cdot \varphi'(t)dt$$

Proof.

Let $F(x) = \int_a^x f(x)dx$. Since $\frac{d}{dt}(F(\varphi(t))) = f(\varphi(t)) \cdot \varphi'(t)$,

$$\int_\alpha^\beta f(\varphi(t)) \cdot \varphi'(t)dt = [F(\varphi(t))]_\alpha^\beta = [F(x)]_a^b = \int_a^b f(x)dx.$$

□

Example

- Note that $\sqrt{1-\sin^2 t} = |\cos t|$ and this is equal to $\cos t$ if $|t| < \frac{\pi}{2}$, hence with $x = \sin t$,

$$\int_0^1 \sqrt{1-x^2}dx = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2}\cos t\, dt = \int_0^{\frac{\pi}{2}} \cos^2 t\, dt$$
$$= \int_0^{\frac{\pi}{2}} \frac{\cos(2t)+1}{2} dt = \left[\frac{\sin(2t)}{4} + \frac{t}{2}\right]_0^{\frac{\pi}{2}}$$
$$= \frac{\pi}{4}.$$

Exercises

- Calculate the indefinite integral. $\int \frac{x}{x^2-1} dx$.
- Calculate the definite integral. $\int_0^1 \frac{1}{x^3-2x^2+x-2} dx$.
- Calculate the indefinite integral. $\int \frac{1}{\cos x} dx$.
- Calculate the indefinite integral. $\int \frac{1}{\cos^2 x+\sin x} dx$.
- Calculate the definite integral. $\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{4-x^2} dx$.