Mathematical Analysis I: Lecture 37

Lecturer: Yoh Tanimoto

23/11/2020

Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 10:00–11:30.
- Office hour: Tuesday 11:30–12:30.

- Basic Mathematics: first few lessons on
  - Tuesday (14:00 – 16:00 CET): Inequalities, Limits and Derivatives
  - Wednesday (14:00 – 16:00 CET): Study of function

  and then upon request.

- Today: Apostol Vol. 1, Chapter 5.

Indefinfite integral of elementary functions

Note that, for $x < 0$, $D(\log|x|) = D(\log(-x)) = -\frac{1}{-x} = \frac{1}{x}$. Altogether,

$f(x)$	$f'(x)$	$\int f(x)dx$	
c (constant)			

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Integration by parts

Recall that, if f, g are differentiable, then it holds that $D(f(x)g(x)) = Df(x)g(x) + f(x)Dg(x)$. By writing this as $Df(x)g(x) = D(f(x)g(x)) - f(x)Dg(x)$, we can find a primitive of $Df(x)g(x)$ if we know a primitive of $f(x)Dg(x)$. Schematically,

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx + C.$$

This is called **integration by parts**.

Integration by parts

Example

- Consider $\int x \cos x dx$.

Integration by parts

Example

- Consider $\int x \cos x dx$. With $f(x) = \sin x, g(x) = x$, this is of the form $f'(x)g(x)$, because $f'(x) = \cos x$.

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$$\int x \cos x dx = x \sin x - \int \sin x \cdot 1 dx + C = x \sin x + \cos x + C.$$

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We can check this results by taking the derivative:
$D(x \sin x + \cos x) = \sin x + x \cos x - \sin x = x \cos x$.

Integration by parts

Example

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Integration by parts

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$$\int \log x dx = x \log x - \int x \cdot \frac{1}{x} dx + C$$
$$= x \log x - \int 1 dx + C = x \log x - x + C.$$

Integration by parts

Example

- Consider $\int x^2 \sin x dx$.

Integration by parts

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$$\begin{aligned}\int x^2 \sin x dx &= x^2(-\cos x) - \int 2x(-\cos x) dx + C \\ &= -x^2 \cos x + 2x \sin x - \int 2 \sin x dx + C \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + C.\end{aligned}$$

Integration by parts

As for indefinite integral, we do not have to find the whole indefinite integral, but we can give values to parts. Let us recall that $f(b) - f(a) = \int_a^b f'(x)dx$.

Lemma

If f, g are differentiable and f', g' are continuous, then

$$\int_a^b f'(x)g(x)dx = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x)dx.$$

Proof.

$(fg)' = f'g + fg'$, hence $\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$ (integration by parts) and the rest goes as follows: If $H(x)$ is a primitive of $h(x)$, then $\int_a^b h(x)dx = H(b) - H(a)$. Note that with $h(x) = f(x)g'(x)$, we have we can take $H(x) = \int_a^x h(x)dx$ and $H(b) - H(a) = \int_a^b h(x)dx - \int_a^a h(x)dx = \int_a^b h(x)dx$. □

Integration by parts

Example

$\int_0^1 xe^{2x}dx$

Integration by parts

Example

$\int_0^1 xe^{2x}dx$
$= \frac{1}{2}[xe^{2x}]_0^1 - \int_0^1 \frac{1}{2}e^{2x}dx = \frac{1}{2}(e^2 - 0) - \frac{1}{4}[e^{2x}]_0^1 = \frac{e^2}{2} - \frac{1}{4}(e^2 - 1) = \frac{e^2}{4} + \frac{1}{4}.$

Substitution

Next, let us consider the case where the integral is of the form $\int \varphi'(x)f'(\varphi(x))dx$. We know that $D(f(\varphi(x))) = \varphi'(x)f'(\varphi(x))$ by the chain rule, hence in this case we have

$$\int \varphi'(x)f'(\varphi(x))dx = f(\varphi(x)) + C.$$

This is called **substitution**.

Substitution

Example

- Consider $\int 2x \sin(x^2) dx$.

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$$\int 2x \sin(x^2)dx = -\cos(x^2) + C.$$

Indeed, by the chain rule,
$D(-\cos(x^2)) = -(2x(-\sin(x^2))) = 2x \sin(x^2)$.

Substitution

Example

- Consider $\int \frac{x}{x^2+1} dx$.

Substitution

Example

- Consider $\int \frac{x}{x^2+1} dx$. Note that $2x = D(x^2)$, and hence

Substitution

Example

- Consider $\int \frac{x}{x^2+1}dx$. Note that $2x = D(x^2)$, and hence

$$\int \frac{x}{x^2+1}dx = \frac{1}{2}\int \frac{2x}{x^2+1}dx = \frac{1}{2}\int \frac{D(x^2)}{x^2+1}dx = \frac{1}{2}\log(x^2+1).$$

Substitution

Example

- Consider $\int \tan x dx$.

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- Consider $\int \tan x dx$. Recall that $\tan x = \frac{\sin x}{\cos x}$ and note that $D(\cos x) = -\sin x$. Hence

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- Consider $\int \tan x dx$. Recall that $\tan x = \frac{\sin x}{\cos x}$ and note that $D(\cos x) = -\sin x$. Hence

$$\int \tan x dx = -\int D(\cos x) \cdot \frac{1}{\cos x} dx + C = -\log|\cos x| + C.$$

Substitution

Lemma

If f, φ are differentiable and f, φ' is continuous, then

$$\int_a^b \varphi'(x)f'(\varphi(x))dx = [f(\varphi(x))]_a^b = [f(y)]_{\varphi(a)}^{\varphi(b)} = f(\varphi(b)) - f(\varphi(a)).$$

Proof.

This follows immediately because $f(\varphi(x))$ is a primitive of $\varphi'(x)f'(\varphi(x))$. □

Substitution

Example

Consider $\int_0^\pi \sin^3 x dx$.

Substitution

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Consider $\int_0^\pi \sin^3 x dx$.

$$\int_0^\pi \sin^3 x dx = -\int_0^\pi (\cos^2 x - 1)\sin x dx$$
$$= \int_0^\pi ((\cos x)^2 - 1)D(\cos x)dx = \left[\frac{\cos^3 x}{3} - \cos x\right]_0^\pi$$
$$= \left(\frac{(-1)^3}{3} - (-1) - (\frac{1^3}{3} - 1)\right) = \frac{4}{3}.$$

Exercises

- Calculate the indefinite integral. $\int xe^x dx$.
- Calculate the indefinite integral. $\int e^x \sin x dx$.
- Calculate the definite integral. $\int_0^1 x^2 e^{-x} dx$.
- Calculate the indefinite integral. $\int x\sqrt{1-x^2} dx$.
- Calculate the indefinite integral. $\int xe^{x^2} dx$.
- Calculate the definite integral. $\int_0^1 x^3 e^{x^2} dx$.