Mathematical Analysis I: Lecture 36

Lecturer: Yoh Tanimoto

20/11/2020

Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 10:00–11:30.
- Office hour: Tuesday 11:30–12:30.

- Basic Mathematics: first few lessons on
  - Tuesday (14:00 – 16:00 CET): Inequalities, Limits and Derivatives
  - Wednesday (14:00 – 16:00 CET): Study of function

  and then upon request.

Exercises

Compute the integral $\int_0^1 x^2 dx$ based on the definition (using the upper and lower sums).

Solution.

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$$\begin{aligned}\overline{S}_I(f, P_n) &= \sum_{j=1}^{n} \left(\frac{j}{n}\right)^2 \cdot \frac{1}{n} \\ &= \frac{1}{n^3} \cdot \sum_{j=1}^{n} j^2 \\ &= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}\end{aligned}$$

Analogously,

$$\underline{S}_I(f, P_n) = \sum_{j=1}^{n} \left(\frac{j-1}{n}\right)^2 \cdot \frac{1}{n} = \frac{1}{n^3} \cdot \sum_{j=1}^{n} (j-1)^2$$

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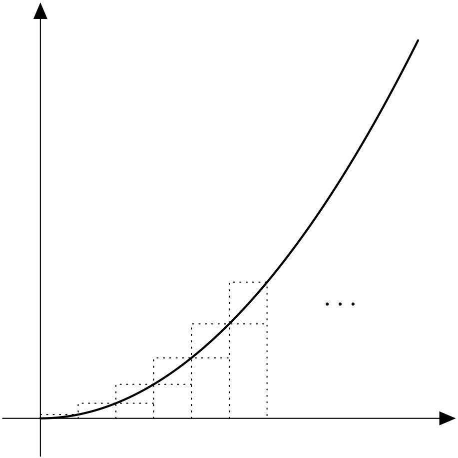

Figure: The upper and lower sum for $f(x) = x^2$.

Compute $\int_{-1}^{1}(x^4 + (x-2)^3 + x(x-1))dx$.
Solution.

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Solution. We have

$$\begin{aligned}
&\int_{-1}^{1}(x^4 + (x-2)^3 + x(x-1))dx \\
&= [\frac{x^5}{5} + \frac{(x-2)^4}{4} + \frac{x^3}{3} - \frac{x^2}{2}]_{-1}^{1} \\
&= (\frac{1}{5} + \frac{1}{4} + \frac{1}{3} - \frac{1}{2}) - (\frac{-1}{5} + \frac{81}{4} + \frac{-1}{3} - \frac{1}{2}) \\
&= \frac{2}{5} - 20 + \frac{2}{3} = \frac{6 - 300 + 10}{15} = -\frac{284}{15}.
\end{aligned}$$

Compute $\int_0^{\frac{\pi}{2}} \sin(2(x+\frac{\pi}{6}))dx$. We have

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \sin(2(x+\frac{\pi}{6}))dx \\
&= [-\frac{1}{2}\cos(2(x+\frac{\pi}{6}))]_0^{\frac{\pi}{2}} \\
&= -\frac{1}{2}(\cos\frac{2\pi}{3} - \cos\frac{\pi}{3}) \\
&= -\frac{1}{2}(-\frac{1}{2} - \frac{1}{2}) = \frac{1}{2}.
\end{aligned}$$

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$$\begin{aligned}
&\int_{-1}^{1} e^{2(x-1)}dx \\
&= [\frac{1}{2}e^{2(x-1)}]_{-1}^{1} \\
&= \frac{1}{2}(e^0 - e^{-4}) \\
&= \frac{1}{2}(1 - e^{-4})
\end{aligned}$$

Compute $\int_1^2 \frac{x^2+3x+1}{x} dx$.
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$$\begin{aligned}
&\int_1^2 \frac{x^2+3x+1}{x} dx \\
&= \int_1^2 (x+3+\frac{1}{x}) dx \\
&= [\frac{x^2}{2} + 3x + \log x]_1^2 \\
&= (\frac{4}{2} + 6 + \log 2) - (\frac{1}{2} + 3 + \log 1) \\
&= \frac{9}{2} + \log 2
\end{aligned}$$

Compute $\int_0^\pi \sin^2 x dx$.
Solution.

Compute $\int_0^\pi \sin^2 x dx$.
Solution. Note that $\cos 2x = 1 - 2\sin^2 x$, hence $\sin^2 x = \frac{1-\cos 2x}{2}$ and

$$\begin{aligned}
&\int_0^\pi \sin^2 x dx \\
&= \int_0^\pi \frac{1-\cos 2x}{2} dx \\
&= \frac{1}{2}[x - \frac{1}{2}\sin 2x]_0^\pi \\
&= \frac{1}{2}((\pi - 0) - (0 - 0)) = \frac{\pi}{2}.
\end{aligned}$$

Compute $\int_0^1 \frac{x^2}{1+x^2} dx$.
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Solution. Note that $D(\arctan x) = \frac{1}{x^2+1}$.

$$
\begin{aligned}
&\int_0^1 \frac{x^2}{1+x^2} dx \\
&= \int_0^1 \frac{x^2+1-1}{1+x^2} dx \\
&= [x - \arctan x]_0^1 \\
&= (1 - \frac{\pi}{4}) - (0-0) = 1 - \frac{\pi}{4}
\end{aligned}
$$

Find the 2nd order Taylor formula for $f(x) = \sqrt{1 + 2x}$ around $x = 0$.
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$f''(x) = 2 \cdot (-\frac{1}{2}\frac{1}{(1+2x)^{\frac{3}{2}}}) = -\frac{1}{(1+2x)^{\frac{3}{2}}}$. Therefore,
$f(x) = 1 + x + \frac{-x^2}{2} + o(x^2)$.

Find the 3nd order Taylor formula for $f(x) = \log(x + 1)$ around $x = 2$.
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Solution. We have $f'(x) = \frac{1}{x+1}$, $f''(x) = -\frac{1}{(x+1)^2}$.
Therefore, $f(x) = \log 3 + \frac{x}{3} - \frac{x^2}{18} + o((x-2)^2)$.

Compute the limit. $\lim_{x \to 0} \frac{x^4}{\cos x - 1 + \frac{x^2}{2}}$.

Solution.

Compute the limit. $\lim_{x\to 0} \frac{x^4}{\cos x - 1 + \frac{x^2}{2}}$.

Solution. We have $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)$, Compute the limit. $\lim_{x\to 0} \frac{x^4}{\cos x - 1 + \frac{x^2}{2}}$.

Solution. We have $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)$, and hence

$$\lim_{x\to 0} \frac{x^4}{\cos x - 1 + \frac{x^2}{2}} = \lim_{x\to 0} \frac{x^4}{\frac{x^4}{24} + o(x^4)} = 24.$$

Determine $\alpha \in \mathbb{R}$ for which the limit $\lim_{x \to 0} \frac{(\sin x)^2 - x^2 + \alpha x^4}{e^{x^6} - 1}$ exists, and in that case, compute the limit.

Solution.

Determine $\alpha \in \mathbb{R}$ for which the limit $\lim_{x\to 0} \frac{(\sin x)^2 - x^2 + \alpha x^4}{e^{x^6} - 1}$ exists, and in that case, compute the limit.

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and hence

$$\frac{(\sin x)^2 - x^2 + \alpha x^4}{e^{x^6} - 1} = \frac{(\alpha - \frac{1}{3})x^4 + \frac{4}{45}x^6 + o(x^6)}{x^6 + o(x^6)}$$

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The limit $x \to 0$ exists if and only if $\alpha - \frac{1}{3} = 0$, Determine $\alpha \in \mathbb{R}$ for which the limit $\lim_{x\to 0} \frac{(\sin x)^2 - x^2 + \alpha x^4}{e^{x^6}-1}$ exists, and in that case, compute the limit.

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