Mathematical Analysis I: Lecture 35

Lecturer: Yoh Tanimoto

19/11/2020

Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 10:00–11:30.
- Office hour: Tuesday 11:30–12:30.

- Today: Apostol Vol. 1, Chapter 5.

Integral calculus

Definition

Let $f : I \to \mathbb{R}$. If there is a function $F : I \to \mathbb{R}$ such that $F' = f$, then F is called a **primitive of** f.

Recall that if there are two primitives F, G of f, then $F(x) - G(x)$ is a constant.
By integrability of continuous functions and Fundamental theorem of calculus 1, there is a primitive if f is continuous: $F(x) = \int_a^x f(t)dt$ is a primitive of f.

Corollary

Let f be a continuous function on a closed bounded interval $I = [a, b]$, and F be a primitive of f. Then $\int_a^b f(t)dt = F(b) - F(a)$.

Proof.

Let $\tilde{F}(x) = \int_a^x f(t)dt$. Then $\tilde{F}$ is a primitive of f, and hence $\tilde{F}(x) - F(x) = c$ (constant). By Fundamental theorem of calculus 2, $\int_a^b f(x)dt = \tilde{F}(b) - \tilde{F}(a) = (F(b) + c) - (F(a) + c) = F(b) - F(a)$. □

This tells us a way to compute the integral $\int_a^b f(x)dx$: we only have to find a primitive $F(x)$ of $f(x)$ and take the difference $F(b) - F(a)$. This is denoted by $[F(x)]_a^b$. Namely,

$$\int_a^b f(x)dx = [F(x)]_a^b.$$

A primitive of f is also written by $\int f(x)dx$ (up to a constant) and it is called the **indefinite integral** of f. With a generic constanti it is written, for example $\int xdx = \frac{x^2}{2} + C$. In contrast, $\int_a^b f(x)dx$ (the integral in the interval $[a, b]$) is called a **definite integral**.

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- Let $f(x) = \sin x$. Then $F(x) = -\cos x$ a primitive of $f(x)$. That is, $\int \sin x dx = -\cos x + C$.
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- Let $f(x) = \frac{1}{x}$. Then $F(x) = \log x$ a primitive of $f(x)$. That is, $\int \frac{1}{x} dx = \log x + C$.
- Let $f(x) = \frac{1}{x^{n+1}}, n \in \mathbb{N}$.

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- Let $f(x) = \frac{1}{x^{n+1}}, n \in \mathbb{N}$. Then $F(x) = -\frac{1}{nx^n}$ a primitive of $f(x)$. That is, $\int \frac{1}{x^{n+1}} dx = -\frac{1}{nx^n} + C$.

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- $\int_1^2 e^x dx = [e^x]_1^2 = e^2 - e$.
- $\int_1^2 \frac{1}{x} dx$

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- $\int_1^2 e^x dx = [e^x]_1^2 = e^2 - e$.
- $\int_1^2 \frac{1}{x} dx = [\log x]_1^2 = \log 2 - \log 1 = \log 2$.
- $\int_1^2 \frac{1}{x^2} dx$

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- $\int_1^2 e^x dx = [e^x]_1^2 = e^2 - e$.
- $\int_1^2 \frac{1}{x} dx = [\log x]_1^2 = \log 2 - \log 1 = \log 2$.
- $\int_1^2 \frac{1}{x^2} dx = [-\frac{1}{x}]_1^2 = -\frac{1}{2} - (-\frac{1}{1}) = \frac{1}{2}$.

Note that $\int_{-1}^2 \frac{1}{x} dx$ is not integrable!

Integral calculus

Lemma

We have the folowing.

- $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx + C.$
- $\int af(x)dx = a\int f(x)dx + C$ *for* $a \in \mathbb{R}$.
- *If* $\int f(x)dx = F(x) + C$, *then* $\int f(x-a)dx = F(x-a) + C$ *for* $a \in \mathbb{R}$.

Proof.

All these follow from the rules for derivatives.

- If $DF(x) = f(x), DG(x) = g(x)$, then $D(F(x) + G(x)) = f(x) + g(x)$.
- If $DF(x) = f(x)$, then $D(aF(x)) = af(x)$.
- If $DF(x) = f(x)$, then $D(F(x-a)) = f(x-a)$ by the chain rule.

□

Lemma

We have the folowing.

- $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x) + C$.
- $\int_a^b cf(x)dx = c \int_a^b f(x)dx + C$ *for* $c \in \mathbb{R}$.

Proof.

This follows immediately from Lemma above. □

With this, we can compute more definite integrals.

Integral calculus

- $\int_0^1 (x-1)^2 dx$

Integral calculus

- $\int_0^1 (x-1)^2 dx = [\frac{(x-1)^3}{3}]_0^1 = \frac{0}{3} - \frac{(-1)^3}{3} = \frac{1}{3}$.
- $\int_{-1}^2 x^2(x-2)dx$

Integral calculus

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- $\int_{-1}^2 x^2(x-2)dx$
  $= \int_{-1}^2 x^3 - 2x^2 dx = [\frac{x^4}{4} - \frac{2x^3}{3}]_{-1}^2 = (4 - \frac{16}{3}) - (\frac{1}{4} - (-\frac{2}{3})) = -\frac{9}{4}$.
- $\int_0^\pi \sin(x - \frac{\pi}{3})dx$

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- $\int_0^\pi \sin(x - \frac{\pi}{3})dx$
  $= [-\cos(x - \frac{\pi}{3})]_0^\pi = -\cos\frac{2\pi}{3} - (-\cos(-\frac{\pi}{3})) = \frac{1}{2} + \frac{1}{2} = 1$.
- $\int_1^2 \frac{2}{x+1} dx$

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- $\int_1^2 \frac{2}{x+1} dx = 2[\log(x+1)]_1^2 = 2(\log 3 - \log 2)$.

Mean value theorem for integral

Theorem

Let f be a continuous function on $I = [a, b]$. Then there is $c \in (a, b)$ such that $\int_a^b f(x)dx = f(c)(b - a)$.

Proof.

Note that $F(x) = \int_a^x f(t)dt$ is differentiable and $F'(x) = f(x)$. By Lagrange's mean value theorem, there is $c \in (a, b)$ such that $F'(c) = \frac{F(b)-F(a)}{b-a}$, that is, $f(c)(b - a) = \int_a^b f(x)dx$. □

Theorem

Let f, g be continuous function on $I = [a, b]$ and assume that $g(x) \geq 0$. Then there is $c \in (a, b)$ such that $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$.

Proof.

As f is continuous, there are m, M such that $m \leq f(x) \leq M$ on I. Since $g(x) \geq 0$, we have $mg(x) \leq f(x)g(x) \leq Mg(x)$. From this, we have

$$m\int_a^b g(x)dx = \int_a^b mg(x)dx \leq \int_a^b f(x)g(x)dx$$
$$= \int_a^b Mg(x)dx = M\int_a^b g(x)dx.$$

This implies that $m \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq M$.

On the other hand, by continuity of f and the intermediate value theorem, there is $c \in (a, b)$ such that $f(c) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}$. □

Some applications

Let us imagine a car travelling at the speed $v(t)$ at time t. Then the distance travelled from time a and b is $\int_a^b v(t)dt$. Indeed, if $x(t)$ is the place of the car at time t, then we have $x'(t) = v(t)$ by definition. By Fundamental theorem of calculus 2, we have $x(b) - x(a) = \int_a^b v(t)dt$.
As another example from physics, consider the situation where someone is pushing up vertically a mass m (kg) to a certain height h. The work done by this motion is, as far as the gravitational force is constant, mgh, where g is the gravitational acceleration. mg is called the weight, which is the downward force. If one is pushing a mass in a changing gravitational field $g(x)$ (like a rocket carrying a payload), the work done by this motion is $\int_{h_1}^{h_2} mg(x)dx$.

Exercises

- Compute $\int_{-1}^{1}(x^4 + (x-2)^3 + x(x-1))dx$.
- Compute $\int_{0}^{\frac{\pi}{2}} \sin(2(x + \frac{\pi}{6}))dx$.
- Compute $\int_{-1}^{1} e^{2(x-1)}dx$.
- Compute $\int_{1}^{2} \frac{x^2+3x+1}{x}dx$.
- Compute $\int_{0}^{\pi} \sin^2 x dx$.
- Compute $\int_{0}^{1} \frac{x^2}{1+x^2}dx$.