Mathematical Analysis I: Lecture 34

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Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 10:00–11:30.
- Office hour: Tuesday 11:30–12:30.

- Today: Apostol Vol. 1, Chapter 3.17-18, 5.

Integrability of continuous functions

Recall that any continuous function f on a closed interval $I = [a, b]$ is uniformly continuous, that is, for given $\epsilon > 0$ there is $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for $x, y \in I$ with $|x - y| < \delta$.

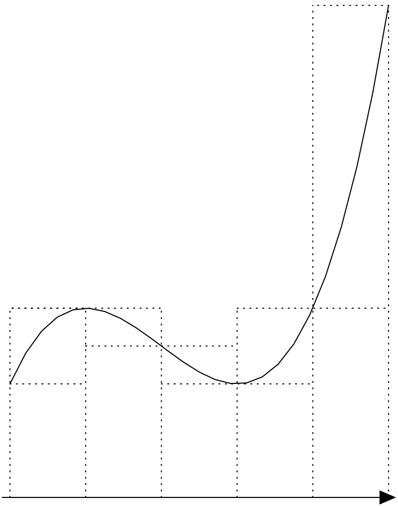

Figure: Integrability of continuous functions on closed bounded intervals.

Integrability of continuous functions

Theorem

Let $I = [a, b]$ be a closed bounded interval. Then any continuous function f on I is integrable.

Proof.

By uniform continuity, for $\frac{\epsilon}{2(b-a)}$ there is δ such that for $x, y \in I, |x - y| < \delta$ it holds that $|f(x) - f(y)| < \frac{\epsilon}{2(b-a)}$. Now, for any partition $P = \{I_j\}_{j=1}^n$ with $\mathrm{diam}\, P = \max\{|I_j| : 1 \leq j \leq n\} < \delta$, we have

$$\overline{S}_I(f, P) - \underline{S}_I(f, P) = \sum_{j=1}^n (\sup_{I_j} f - \inf_{I_j} f)|I_j| < \sum_{j=1}^n \frac{\epsilon}{2(b-a)}|I_j| = \frac{\epsilon}{2} < \epsilon.$$

Therefore, f is integrable. □

Integrability of continuous functions

Example

(of a nonintegrable function)

If $f(x) = \begin{cases} 0 & x \text{ rational} \\ 1 & x \text{ irrational} \end{cases}$, $f(x)$ is not integrable:
$\overline{S}_I(f, P) = 1, \underline{S}_I(f, P) = 0$, because any interval I contains both a rational number and an irrational number.

Some properties of integral

Theorem

Let I be a bounded interval, f, g bounded and integrable on I.

- *If $c, d \in \mathbb{R}$, then $cf + dg$ is integrable on I and* $\int_I (cf(t) + dg(t))dt = c\int_I f(t)dt + d\int_I g(t)dt$.
- *If $f \leq g$, then* $\int_I f(t)dt \leq \int_I g(t)dt$.
- *If $\underline{I} \subset I$, then f is integrable on $\underline{I}$. If $P = \{I_j : 1 \leq j \leq n\}$ is a partition of I, then* $\int_I f(t)dt = \sum_{j=1}^n \int_{I_j} f(t)dt$.

Proof.

- Integrability of cf is easy: $\underline{S}_I(cf, P) = c\underline{S}_I(f, P)$ if $c \geq 0$ and $\underline{S}_I(cf, P) = c\overline{S}_I(f, P)$ if $c < 0$. If $c = 0$ everything becomes 0, and if $c > 0$ one obtains the limit directly. If $c < 0$ sup and inf are exchanged.

  Let f, g be integrable. We have $\inf_{I_j} f + \inf_{I_j} g \leq \inf_{I_j}(f + g)$, hence for any partition P, $\underline{S}_I(f, P) + \underline{S}_I(g, P) \leq \underline{S}_I(f + g, P)$. Analogously, $\overline{S}_I(f + g, P) \leq \overline{S}_I(f, P) + \overline{S}_I(g, P)$. By taking inf and sup with respect to P, we obtain integrability of $f + g$ and the equality $\int_I (cf(t) + df(t))dt = c \int_I f(t)dt + d \int_I g(t)dt$.
- If $f \leq g$, then $\overline{S}_I(f, P) \leq \overline{S}_I(g, P)$ for any P. Similarly $\underline{S}_I(f, P) \leq \underline{S}_I(g, P)$.

Proof.

- Let $\underline{I} \subset I$. Then, for any partition P of I, we can take a refinement P' which consists of the intervals of the form $I_j \cap \underline{I}$ and $I_j \setminus \underline{I}$ (the latter may be a union of two intervals).
  Let $P_2 = \{I_j \cap \underline{I} : 1 \le j \le n\}$. Then

$$\begin{aligned}
&\overline{S}_{\underline{I}}(f, P_2) - \underline{S}_{\underline{I}}(f, P_2) \\
&\le \overline{S}_{\underline{I}}(f, P_2) - \underline{S}_{\underline{I}}(f, P_2) + \sum_{j=1}^{n} (\sup_{I_j \setminus \underline{I}} f - \inf_{I_j \setminus \underline{I}} f)|I_j \setminus \underline{I}| \\
&= \overline{S}_{\underline{I}}(f, P') - \underline{S}_{\underline{I}}(f, P')
\end{aligned}$$

  As f is integrable, there is P' such that $\overline{S}_{\underline{I}}(f, P') - \underline{S}_{\underline{I}}(f, P') < \epsilon$, then $\overline{S}_{\underline{I}}(f, P_2) - \underline{S}_{\underline{I}}(f, P_2) < \epsilon$.

Proof.

Let us consider the case $n = 2$, that is $I = I_1 \cup I_2$. For P_1, P_2 partitions of I_1, I_2, $P = P_1 \cup P_2$ is a partition of I. Then

$$\begin{aligned} \underline{S}_{I_1}(f, P_1) + \underline{S}_{I_2}(f, P_2) &= \underline{S}_I(f, P_1 \cup P_2) \le \overline{S}_I(f, P_1 \cup P_2) \\ &= \overline{S}_{I_1}(f, P_1) + \overline{S}_{I_2}(f, P_2). \end{aligned}$$

If $\overline{S}_{I_1}(f, P_1) - \underline{S}_{I_1}(f, P_1) < \frac{\varepsilon}{2}$ and $\overline{S}_{I_2}(f, P_2) - \underline{S}_{I_2}(f, P_2) < \frac{\varepsilon}{2}$, then $\overline{S}_I(f, P_1 \cup P_2) - \underline{S}_I(f, P_1 \cup P_2) < \varepsilon$ and the limits coincide: $\int_I f(x)dx = \int_{I_1} f(x)dx + \int_{I_2} f(x)dx$. □

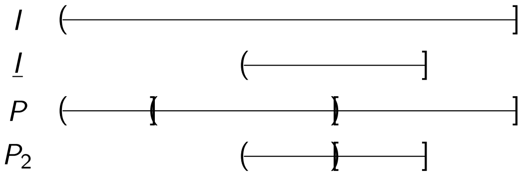

Figure: The patition P_2 of a subinterval $\underline{I}$ of I obtained from a partition P of I.

Integrability of continuous functions

Corollary

If f is continuous, then $|f|$ is continuous and $|\int_I f(x)dx| \leq \int_I |f(x)|dx$.

Proof.

This follows from $-|f(x)| \leq f(x) \leq |f(x)|$. □

Some properties of integral

Definition

If $a < b$, then we put $\int_b^a f(x)dx = -\int_a^b f(x)dx$.

Lemma

It holds that $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ *for all* $a, b, c \in \mathbb{R}$.

Proof.

If $a < c < b$, then this follows from 3(iii). If $a < b < c$, then

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^b f(x)dx - \int_c^b f(x)dx.$$

The other cases are analogous. □

Integration and differentiation

Theorem (Fundamental theorem of calculus 1)

Let $I = [a, b]$ a bounded closed interval and $f : I \to \mathbb{R}$ continuous. Then the function of x on I defined by

$$F(x) = \int_a^x f(t)dt$$

is differentiable and $F'(x) = f(x)$.

Proof.

We have

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right) = \frac{1}{h}\int_x^{x+h} f(t)dt$$

As f is continuous, $\left|\int_x^{x+h} f(t)dt - f(x)h\right| = \left|\int_x^{x+h} f(t)dt - \int_x^{x+h} f(x)dt\right| \le \int_x^{x+h} |f(t)-f(x)|dt$ and for any $\epsilon > 0$ there is $\delta > 0$ such that if $|t-x| < \delta$, then $|f(t)-f(x)| < \epsilon$, therefore, $\int_x^{x+h} |f(t)-f(x)|dt < h\epsilon$. Then, for such h, $\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| = \left|\frac{1}{h}\int_x^{x+h} f(t)dt - f(x)\right| < \epsilon$. Since $\epsilon > 0$ is arbitrary, $\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = f(x)$. □

Theorem (Fundamental theorem of calculus 2)

Let $I = [a, b]$ be a closed bounded interval and $f : I \to \mathbb{R}$ differentiable in (a, b) and f' is continuous and extends to a continuous function on Then

$$f(x) - f(a) = \int_a^x f'(t)dt.$$

Proof.

$D(f(x) - f(a)) = f'(x)$, while $D(\int_a^x f'(t)dt) = f'(x)$ by the previous theorem. Therefore, $D(f(x) - f(a) - \int_a^x f'(t)dt) = 0$, and we know that $f(x) - f(a) - \int_a^x f'(t)dt$ is constant, but with $x = a$, $f(a) - f(a) - \int_a^a f'(t)dt = 0$, hence $f(x) - f(a) - \int_a^x f'(t)dt = 0$. □

Integration and differentiation

Theorem 8 allows us to compute integrals of certain functions.

- We know that $D(x^{n+1}) = (n+1)x^n$, or $D(\frac{x^{n+1}}{n+1}) = x^n$. Hence $\int_a^x t^n dt = \frac{1}{n+1}(x^{n+1} - a^{n+1})$. We know that $D(e^x) = e^x$, hence
- $\int_a^x e^t dt = (e^x - e^a)$.

Exercises

will be proposed tomorrow.
Do you have requests on the topics for exercises?