Mathematical Analysis I: Lecture 33

Lecturer: Yoh Tanimoto

16/11/2020

Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 10:00–11:30.
- Office hour: Tuesday 11:30–12:30.

- Today: Apostol Vol. 1, Chapter 1.9-17.

Integral

Given a function f, we consider (Riemann) integral. This is a concept that extends the area of familiar figures such as triangles and disks. If $f(t)$ represents the velocity of a car at time t, then the integral of f gives the distance the car travels in a time interval. If f is the density of a piece of iron, the integral gives the weight.
The area of a region defined by a function can be approximated by rectangles. We know that the area of a rectangle with sides a, b is ab.

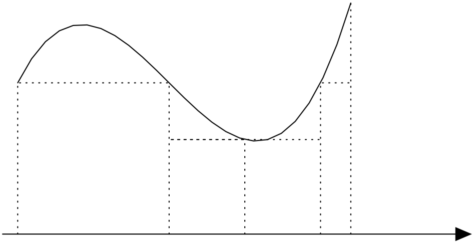


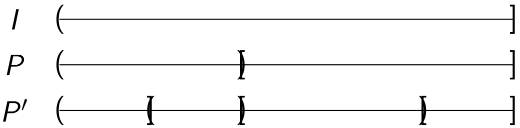
Figure: Approximating the area surrounded by $f(x)$ by rectangles.

Integral

For an interval $I = (a, b)$ or $(a, b]$ etc., we define $|I| = b - a$.

Definition

- Let I be a bounded interval in $\mathbb{R}$. A partition of I is a finite set of disjoint intervals $P = \{I_j : 1 \leq j \leq n\}$ such that $\bigcup_{j=1}^n I_j = I$.
- $\mathrm{diam}(P) = \max\{|I_j| : 1 \leq j \leq n\}$.
- A partition P' is called a refinement of P if every interval of P admits a partizione formed by intervals in P'. That is, every $I_j \in P$ can be written as $I_j = \bigcup_{k=1}^{n_j} I_{jk}, I_{jk} \in P'$.
  We denote this by $P' \succ P$.
- If P, P' are partitions of I, we define $P \wedge P' = \{I \cap I' : I \in P, I \in P', I \cap I' \neq \emptyset\}$. We have $P \wedge P' \succ P, P'$.



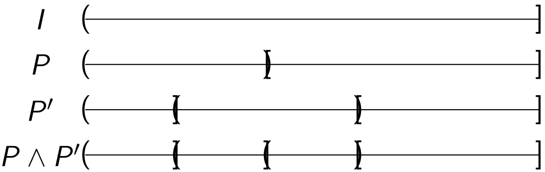


Figure: Above: a partition P of an interval I and a refinement $P' \succ P$. Below: two partitions P, P' of I and $P \wedge P'$.

If P is a partition of I, then $|I| = \sum_{j=1}^{n} |I_j|$.

Definition

For a partition P of I and a bounded function $f : I \to \mathbb{R}$, we put

$$\underline{S}_I(f, P) := \sum_{j=1}^{n} (\inf_{I_j} f)|I_j|, \ \text{(“lower sum”)}$$

$$\overline{S}_I(f, P) := \sum_{j=1}^{n} (\sup_{I_j} f)|I_j| \ \text{(“upper sum”)}.$$

We have $\underline{S}_I(f, P) \le \overline{S}_I(f, P)$.

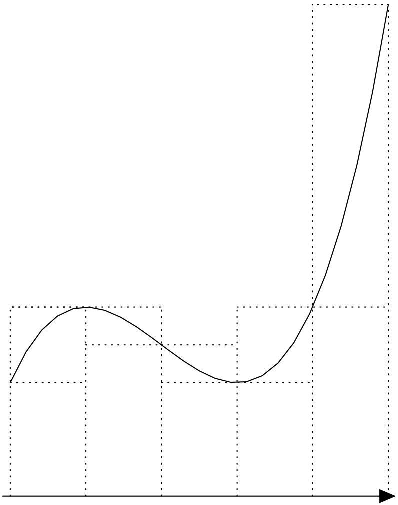


Figure: $\overline{S}_I(f, P)$ and $\underline{S}_I(f, P)$ for a given a partition P of I.

Integral

Example

$I = [0,1]$. $P_n = \{[0, \frac{1}{n}), [\frac{1}{n}, \frac{2}{n}), \cdots, [\frac{n-1}{n}, 1]\}$. If $n' = mn$ for $m \in \mathbb{N}$, then $P_{n'} \succ P_n$. $\mathrm{diam}(P_n) = \frac{1}{n}$.

- If $f(x) = a$, then $\underline{S}_I(f, P_n) = \overline{S}_I(f, P_n) = a$.

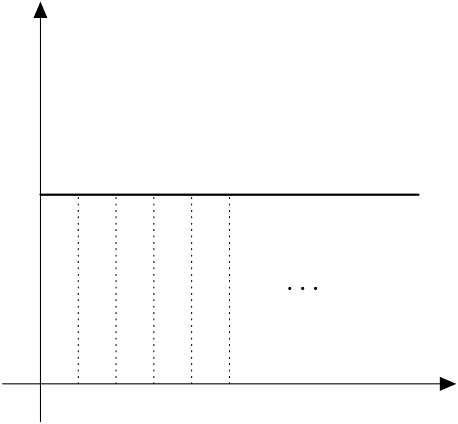


Figure: The upper and lower sum for $f(x) = a$ (constant).

Example

$I = [0,1]$. $P_n = \{[0, \frac{1}{n}), [\frac{1}{n}, \frac{2}{n}), \cdots, [\frac{n-1}{n}, 1]\}$. If $n' = mn$ for $m \in \mathbb{N}$, then $P_{n'} \succ P_n$. $\mathrm{diam}(P) = \frac{1}{n}$.

- Let $f(x) = x$.

$$\overline{S}_I(f, P_n) = \sum_{j=1}^{n} \frac{j}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \frac{n(n+1)}{2}$$

Analogously,

$$\underline{S}_I(f, P_n) = \sum_{j=1}^{n} \frac{j-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \left(\frac{n(n+1)}{2} - n\right) = \frac{1}{n^2} \frac{n(n-1)}{2}.$$

Therefore, by taking $n \to \infty$, we obtain $\lim_{n\to\infty} \overline{S}_I(f, P_n) = \lim_{n\to\infty} \underline{S}_I(f, P_n) = \frac{1}{2}$, which is the area of the triangle $\{(x, y) : x \in [0,1], 0 \le y \le x = f(x)\}$.

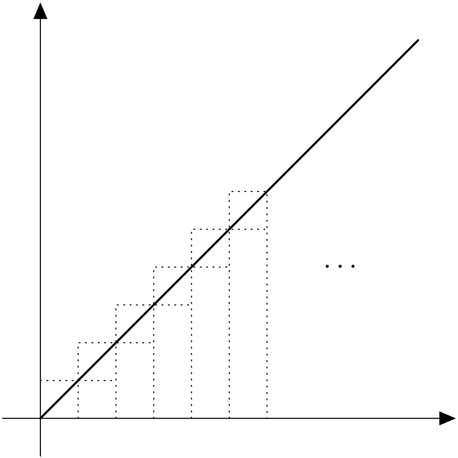


Figure: The upper and lower sum for $f(x) = x$.

Integral

Lemma

Let I be a bounded interval, $f : I \to \mathbb{R}$ a bounded function. If P, P' are two partitions of I, then

1. *If $P \prec P'$, then*

$$\underline{S}_I(f, P) \leq \underline{S}_I(f, P') \leq \overline{S}_I(f, P') \leq \overline{S}_I(f, P).$$

2. $\underline{S}_I(f, P) \leq \overline{S}_I(f, P')$

Proof.

1. If $P \prec P'$, then we can take $P = \{I_j : 1 \le j \le n\}$ and $P' = \{I_{jk} : 1 \le j \le n, 1 \le j \le n_j\}$ such that $\cup_{k=1}^{n_j} I_{jk} = I_j$. Then, for every j, k, $\inf_{I_j} f \le \inf_{I_{jk}} f$, $\sup_{I_j} f \ge \sup_{I_{jk}} f$. It follows that

$$\underline{S}_I(f, P) = \sum_{j=1}^{n} (\inf_{I_j} f)|I_j| = \sum_{j=1}^{n} (\inf_{I_j} f) \sum_{k=1}^{n_k} |I_{jk}| = \sum_{j=1}^{n} \sum_{k=1}^{n_k} (\inf_{I_j} f)|I_{jk}|$$

$$\le \sum_{j=1}^{n} \sum_{k=1}^{n_k} (\inf_{I_{jk}} f)|I_{jk}| = \underline{S}_I(f, P').$$

$$\overline{S}_I(f, P) = \sum_{j=1}^{n} (\sup_{I_j} f)|I_j| = \sum_{j=1}^{n} (\sup_{I_j} f) \sum_{k=1}^{n_k} |I_{jk}| = \sum_{j=1}^{n} \sum_{k=1}^{n_k} (\sup_{I_j} f)|I_{jk}|$$

$$\ge \sum_{j=1}^{n} (\sup_{I_{jk}} f) \sum_{k=1}^{n_k} |I_{jk}| = \overline{S}_I(f, P').$$

Note that $\underline{S}_I(f, Q) \le \overline{S}_I(f, Q)$ for any partition Q.

Proof.

(ii) Since $P \prec P \wedge P', P' \prec P \wedge P'$, it follows from the previous point $\underline{S}_I(f, P) \leq \underline{S}_I(f, P \wedge P') \leq \overline{S}_I(f, P \wedge P') \leq \overline{S}_I(f, P')$.

□

Integral

Definition

Let $I = (a, b)$ or $[a, b]$ etc. a bounded interval and $f : I \to \mathbb{R}$ bounded. f is said to be **integrable on** I if

$$\underbrace{\sup_P \underline{S}_I(f, P)}_{\text{"lower integral"}} = \underbrace{\inf_P \overline{S}_I(f, P)}_{\text{"upper integral"}},$$

where $\inf_P$ and $\sup_P$ are taken over all possible partitions of P of I and in this case we denote this number by

$$\int_I f(x)dx = \int_a^b f(x)dx.$$

x does not have any meaning, and one can also write thsi as $\int_I f(t)dt$.

Integral

Example

- $\int_0^1 a dx = a$. Indeed, for all partitions $\underline{S}_I(f, P) = \overline{S}_I(f, P) = a$.
- $\int_0^1 x dx = \frac{1}{2}$. Indeed, with $f(x) = x$, we have found P_n such that $\underline{S}_I(f, P) = \frac{n(n-1)}{2n^2}, \overline{S}_I(f, P) = \frac{n(n+1)}{2n^2}$, hence the sup and the inf coincide and it is $\frac{1}{2}$.

In general, it is difficult to show integrability by definition. Fortunately, we can prove that continuous functions on a closed bounded interval are integrable, and we also have the fundamental theorems of calculus, that let us calculate integrals with the knowledge of derivatives.

Exercises

- Compute the integral $\int_0^1 x^2 dx$ based on the definition.