Mathematical Analysis I: Lecture 32

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Exercises

Compute the limit. $\lim_{x \to 0} \frac{\sin^2 x}{x^2}$.
Solution.

Compute the limit. $\lim_{x\to 0} \frac{\sin^2 x}{x^2}$.
Solution. We have $D(\sin^2 x)$

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Solution. We have $D(\sin^2 x) = 2\sin x\cos x$,

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Solution. We have $D(\sin^2 x) = 2 \sin x \cos x$, $D(x^2) = 2x$,

Compute the limit. $\lim_{x\to 0} \frac{\sin^2 x}{x^2}$.
Solution. We have $D(\sin^2 x) = 2\sin x \cos x$, $D(x^2) = 2x$, and hence $\lim_{x\to 0} \frac{\sin^2 x}{x^2} = \lim_{x\to 0} \frac{2\sin x \cos x}{2x} = 1$.

Compute the limit. $\lim_{x \to 0} \frac{\sin x - x}{x^3}$.
Solution.

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Solution. We have $D(\sin x - x) = \cos x - 1$,

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Solution. We have $D(\sin x - x) = \cos x - 1$, $D(x^3) = 3x^2$,

Compute the limit. $\lim_{x \to 0} \frac{\sin x - x}{x^3}$.
Solution. We have $D(\sin x - x) = \cos x - 1$, $D(x^3) = 3x^2$, and further $D(\cos x - 1)$

Compute the limit. $\lim_{x \to 0} \frac{\sin x - x}{x^3}$.
Solution. We have $D(\sin x - x) = \cos x - 1$, $D(x^3) = 3x^2$, and further $D(\cos x - 1) = -\sin x$,

Compute the limit. $\lim_{x\to 0}\frac{\sin x - x}{x^3}$.
Solution. We have $D(\sin x - x) = \cos x - 1$, $D(x^3) = 3x^2$, and further $D(\cos x - 1) = -\sin x$, $D(3x^2) = 6x$

Compute the limit. $\lim_{x\to 0} \frac{\sin x - x}{x^3}$.
Solution. We have $D(\sin x - x) = \cos x - 1$, $D(x^3) = 3x^2$, and further $D(\cos x - 1) = -\sin x$, $D(3x^2) = 6x$ hence

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{\cos -1}{3x^2} = \lim_{x\to 0} \frac{-\sin x}{6x} = -\frac{1}{6}.$$

Compute the limit. $\lim_{x\to\infty} \frac{\log(x^3+1)}{\log x}$.

Solution.

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Solution. We have $D(\log(x^3+1))$

Compute the limit. $\lim_{x \to \infty} \frac{\log(x^3+1)}{\log x}$.
Solution. We have $D(\log(x^3+1)) = \frac{3x^2}{x^3+1}$,

Compute the limit. $\lim_{x\to\infty} \frac{\log(x^3+1)}{\log x}$.

Solution. We have $D(\log(x^3+1)) = \frac{3x^2}{x^3+1}$, $D(\log x) = \frac{1}{x}$ and hence

Compute the limit. $\lim_{x\to\infty} \frac{\log(x^3+1)}{\log x}$.

Solution. We have $D(\log(x^3+1)) = \frac{3x^2}{x^3+1}$, $D(\log x) = \frac{1}{x}$ and hence

$$\lim_{x\to\infty} \frac{\log(x^3+1)}{\log x} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^3+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{3x^3}{x^3+1} = 3.$$

Compute the limit. $\lim_{x \to 0} x \log x$.
Solution.

Compute the limit. $\lim_{x \to 0} x \log x$.
Solution. We have $\lim_{x \to 0} x \log x = \lim_{x \to 0} \frac{\log x}{\frac{1}{x}}$

Compute the limit. $\lim_{x\to 0} x \log x$.

Solution. We have $\lim_{x\to 0} x \log x = \lim_{x\to 0} \frac{\log x}{\frac{1}{x}}$ and $D(\log x)$

Compute the limit. $\lim_{x\to 0} x \log x$.
Solution. We have $\lim_{x\to 0} x \log x = \lim_{x\to 0} \frac{\log x}{\frac{1}{x}}$ and $D(\log x) = \frac{1}{x}$,

Compute the limit. $\lim_{x\to 0} x\log x$.
Solution. We have $\lim_{x\to 0} x\log x = \lim_{x\to 0} \frac{\log x}{\frac{1}{x}}$ and $D(\log x) = \frac{1}{x}$, $D(\frac{1}{x}) = \frac{-1}{x^2}$ and hence

Compute the limit. $\lim_{x\to 0} x\log x$.

Solution. We have $\lim_{x\to 0} x\log x = \lim_{x\to 0} \frac{\log x}{\frac{1}{x}}$ and $D(\log x) = \frac{1}{x}$, $D(\frac{1}{x}) = \frac{-1}{x^2}$ and hence $\lim_{x\to 0} x\log x = \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0}(-x) = 0$.

Find the second order Taylor formula. $f(x) = \sin(x^2)$ as $x \to 0$.
Solution.

Find the second order Taylor formula. $f(x) = \sin(x^2)$ as $x \to 0$.
Solution. We have $f'(x) = 2x\cos(x^2)$

Find the second order Taylor formula. $f(x) = \sin(x^2)$ as $x \to 0$.
Solution. We have $f'(x) = 2x\cos(x^2)$, $f''(x) = 2\cos(x^2) - 4x^2\sin(x^2)$

Find the second order Taylor formula. $f(x) = \sin(x^2)$ as $x \to 0$.
Solution. We have $f'(x) = 2x\cos(x^2)$, $f''(x) = 2\cos(x^2) - 4x^2\sin(x^2)$ and $f(0) = 0, f'(0) = 0, f''(0) = 2$,

Find the second order Taylor formula. $f(x) = \sin(x^2)$ as $x \to 0$.
Solution. We have $f'(x) = 2x\cos(x^2)$, $f''(x) = 2\cos(x^2) - 4x^2\sin(x^2)$ and $f(0) = 0, f'(0) = 0, f''(0) = 2$, and hence
$f(x) = 0 + 0x + \frac{2x^2}{2!} + o(x^2) = x^2 + o(x^2)$ as $x \to 0$.

Find the second order Taylor formula. $f(x) = \sqrt{x^2 + 1}$ as $x \to 1$.

Solution.

Find the second order Taylor formula. $f(x) = \sqrt{x^2+1}$ as $x \to 1$.

Solution. We have $f'(x) = \frac{x}{\sqrt{x^2+1}}$

Find the second order Taylor formula. $f(x) = \sqrt{x^2 + 1}$ as $x \to 1$.

Solution. We have $f'(x) = \frac{x}{\sqrt{x^2+1}}$, $f''(x) = \frac{\sqrt{x^2+1} - x\frac{x}{\sqrt{x^2+1}}}{x^2+1} = \frac{1}{(x^2+1)^{\frac{3}{2}}}$

Find the second order Taylor formula. $f(x) = \sqrt{x^2+1}$ as $x \to 1$.

Solution. We have $f'(x) = \frac{x}{\sqrt{x^2+1}}$, $f''(x) = \frac{\sqrt{x^2+1} - x\frac{x}{\sqrt{x^2+1}}}{x^2+1} = \frac{1}{(x^2+1)^{\frac{3}{2}}}$ and hence $f(1) = \sqrt{2}, f'(1) = \frac{1}{\sqrt{2}}, f''(1) = \frac{1}{2^{\frac{3}{2}}}$.

Find the second order Taylor formula. $f(x) = \sqrt{x^2+1}$ as $x \to 1$.

Solution. We have $f'(x) = \frac{x}{\sqrt{x^2+1}}$, $f''(x) = \frac{\sqrt{x^2+1} - x\frac{x}{\sqrt{x^2+1}}}{x^2+1} = \frac{1}{(x^2+1)^{\frac{3}{2}}}$ and hence $f(1) = \sqrt{2}, f'(1) = \frac{1}{\sqrt{2}}, f''(1) = \frac{1}{2^{\frac{3}{2}}}$. Therefore,

$f(x) = \sqrt{2} + \frac{(x-1)}{\sqrt{2}} + \frac{(x-1)^2}{2^{\frac{5}{2}}} + o((x-1)^2)$ as $x \to 1$.

Find the second order Taylor formula. $f(x) = \sin(x) - 1$ as $x \to \frac{\pi}{2}$.
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Find the second order Taylor formula. $f(x) = \sin(x) - 1$ as $x \to \frac{\pi}{2}$.
Solution. We have $f'(x) = \cos x$, $f''(x) = -\sin x$

Find the second order Taylor formula. $f(x) = \sin(x) - 1$ as $x \to \frac{\pi}{2}$.
Solution. We have $f'(x) = \cos x$, $f''(x) = -\sin x$ and hence $f(\frac{\pi}{2}) = 0, f'(\frac{\pi}{2}) = 0, f''(\frac{\pi}{2}) = -1$.

Find the second order Taylor formula. $f(x) = \sin(x) - 1$ as $x \to \frac{\pi}{2}$.
Solution. We have $f'(x) = \cos x$, $f''(x) = -\sin x$ and hence
$f(\frac{\pi}{2}) = 0, f'(\frac{\pi}{2}) = 0, f''(\frac{\pi}{2}) = -1$. Therefore,
$f(x) = -\frac{(x-\frac{\pi}{2})^2}{2} + o((x - \frac{\pi}{2})^2)$ as $x \to \frac{\pi}{2}$.

Find the second order Taylor formula. $f(x) = \frac{e^x - 1}{\cos x}$ as $x \to 0$.

Solution.

Find the second order Taylor formula. $f(x) = \frac{e^x - 1}{\cos x}$ as $x \to 0$.

Solution. We have $f'(x) = \frac{e^x \cos x + (e^x - 1)\sin x}{\cos^2 x}$,

Find the second order Taylor formula. $f(x) = \frac{e^x - 1}{\cos x}$ as $x \to 0$.
Solution. We have $f'(x) = \frac{e^x \cos x + (e^x - 1)\sin x}{\cos^2 x}$,

$$
\begin{aligned}
&f''(x) \\
&= \frac{(e^x(\cos x - \sin x) + (e^x \sin x + (e^x - 1)\cos x))\cos^2 x}{\cos^4 x} \\
&\quad - \frac{(e^x \cos x + (e^x - 1)\sin x)(-2\sin x \cos x)}{\cos^4 x} \\
&= \frac{(2e^x - 1)\cos^3 x + 2e^x \sin x \cos^2 x + 2e^x \sin^2 x \cos x - 2\sin^2 x \cos x}{\cos^4 x}
\end{aligned}
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Find the second order Taylor formula. $f(x) = \frac{e^x - 1}{\cos x}$ as $x \to 0$.
Solution. We have $f'(x) = \frac{e^x \cos x + (e^x - 1)\sin x}{\cos^2 x}$,

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and hence $f(0) = 0, f'(0) = 1, f''(0) = 1$.

Find the second order Taylor formula. $f(x) = \frac{e^x - 1}{\cos x}$ as $x \to 0$.

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\end{aligned}
$$

and hence $f(0) = 0, f'(0) = 1, f''(0) = 1$. Therefore, $f(x) = x + \frac{x^2}{2} + o(x^2)$ as $x \to 0$.

Find the n-th order Taylor formula. $f(x) = \cos(x)$ as $x \to 0$.
Solution.

Find the n-th order Taylor formula. $f(x) = \cos(x)$ as $x \to 0$.
Solution. $f^{(4n)}(x) = \cos x, f^{(4n+1)}(x) = -\sin x, f^{(4n+2)}(x) = -\cos x, f^{(4n+3)}(x) = \sin x,$

Find the n-th order Taylor formula. $f(x) = \cos(x)$ as $x \to 0$.
Solution. $f^{(4n)}(x) = \cos x, f^{(4n+1)}(x) = -\sin x, f^{(4n+2)}(x) = -\cos x, f^{(4n+3)}(x) = \sin x$, and hence
$f^{(4n)}(0) = 1, f^{(4n+1)}(0) = 0, f^{(4n+2)}(0) = -1, f^{(4n+3)}(0) = 0$, and

Find the n-th order Taylor formula. $f(x) = \cos(x)$ as $x \to 0$.
Solution. $f^{(4n)}(x) = \cos x, f^{(4n+1)}(x) = -\sin x, f^{(4n+2)}(x) = -\cos x, f^{(4n+3)}(x) = \sin x$, and hence
$f^{(4n)}(0) = 1, f^{(4n+1)}(0) = 0, f^{(4n+2)}(0) = -1, f^{(4n+3)}(0) = 0$, and

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + o(x^{2n}) \\ &= \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} + o(x^{2n}).\end{aligned}$$

Find the n-th order Taylor formula. $f(x) = \log(1 + x)$ as $x \to 0$.
Solution.

Find the n-th order Taylor formula. $f(x) = \log(1+x)$ as $x \to 0$.

Solution. $f^{(2n)}(x) = \frac{-(2n-1)!}{(1+x)^{2n}}, f^{(2n+1)}(x) = \frac{(2n)!}{(1+x)^{2n+1}}$, and

Find the n-th order Taylor formula. $f(x) = \log(1+x)$ as $x \to 0$.
Solution. $f^{(2n)}(x) = \frac{-(2n-1)!}{(1+x)^{2n}}, f^{(2n+1)}(x) = \frac{(2n)!}{(1+x)^{2n+1}}$, and
$f^{(2n)}(0) = (2n-1)!, f^{(2n+1)}(0) = -(2n)!$, and

Find the n-th order Taylor formula. $f(x) = \log(1+x)$ as $x \to 0$.
Solution. $f^{(2n)}(x) = \frac{-(2n-1)!}{(1+x)^{2n}}, f^{(2n+1)}(x) = \frac{(2n)!}{(1+x)^{2n+1}}$, and
$f^{(2n)}(0) = (2n-1)!, f^{(2n+1)}(0) = -(2n)!$, and

$$\begin{aligned}\log(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1}x^n}{n} + o(x^n) \\ &= \sum_{k=0}^{n} \frac{(-1)^{k+1}x^k}{k} + o(x^n).\end{aligned}$$

Find the n-th order Taylor formula. $f(x) = \sin(x^2)$ as $x \to 0$.

Solution.

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Solution. We know $\sin y = \sum_{k=0}^{n} \frac{(-1)^k y^{2k+1}}{(2k+1)!} + o(y^{2n+1})$ as $y \to 0$

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Compute the limit.

$$\lim_{x \to 0} \frac{e^x + \cos(x) - \sin(x) - 2}{\tan(2x^3)}.$$

Solution.

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Solution. As $x \to x_0 = 0$, we have

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$$\lim_{x \to 0} \frac{e^x + \cos(x) - \sin(x) - 2}{\tan(2x^3)}.$$

Solution. As $x \to x_0 = 0$, we have

- $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$

Compute the limit.

$$\lim_{x \to 0} \frac{e^x + \cos(x) - \sin(x) - 2}{\tan(2x^3)}.$$

Solution. As $x \to x_0 = 0$, we have

- $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$
- $\cos x = 1 - \frac{x^2}{2} + o(x^3)$

Compute the limit.

$$\lim_{x\to 0} \frac{e^x + \cos(x) - \sin(x) - 2}{\tan(2x^3)}.$$

Solution. As $x \to x_0 = 0$, we have

- $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$
- $\cos x = 1 - \frac{x^2}{2} + o(x^3)$
- $\sin x = x - \frac{x^3}{6} + o(x^3)$

Compute the limit.

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Solution. As $x \to x_0 = 0$, we have

- $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$
- $\cos x = 1 - \frac{x^2}{2} + o(x^3)$
- $\sin x = x - \frac{x^3}{6} + o(x^3)$
- $\tan(2x^3) = 2x^3 + o(x^3)$

Compute the limit.

$$\lim_{x \to 0} \frac{e^x + \cos(x) - \sin(x) - 2}{\tan(2x^3)}.$$

Solution. As $x \to x_0 = 0$, we have

- $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$
- $\cos x = 1 - \frac{x^2}{2} + o(x^3)$
- $\sin x = x - \frac{x^3}{6} + o(x^3)$
- $\tan(2x^3) = 2x^3 + o(x^3)$

Then it holds, as $x \to 0$,

$$\begin{aligned} &\frac{e^x + \cos(x) - \sin(x) - 2}{\tan(2x^3)} \\ &= \frac{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + 1 - \frac{x^2}{2} - x + \frac{x^3}{6} - 2 + o(x^3)}{2x^3 + o(x^3)} \\ &= \frac{\frac{x^3}{3} + o(x^3)}{2x^3 + o(x^3)} \end{aligned}$$

Compute the limit.

$$\lim_{x\to 0} \frac{e^x + \cos(x) - \sin(x) - 2}{\tan(2x^3)}.$$

Solution. As $x \to x_0 = 0$, we have

- $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$
- $\cos x = 1 - \frac{x^2}{2} + o(x^3)$
- $\sin x = x - \frac{x^3}{6} + o(x^3)$
- $\tan(2x^3) = 2x^3 + o(x^3)$

Then it holds, as $x \to 0$,

$$\begin{aligned}
&\frac{e^x + \cos(x) - \sin(x) - 2}{\tan(2x^3)} \\
&= \frac{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + 1 - \frac{x^2}{2} - x + \frac{x^3}{6} - 2 + o(x^3)}{2x^3 + o(x^3)} \\
&= \frac{\frac{x^3}{3} + o(x^3)}{2x^3 + o(x^3)}
\end{aligned}$$

hence $\lim_{x\to 0} \frac{x - \ln(1-x) - 2x\sqrt{1+x}}{\sin(x) - xe^x} = \frac{1}{6}$.

For which α does the following limit exist?

$$\lim_{x\to 0} \frac{\left(\frac{1+x^2}{1-x^2}\right) - \alpha \sin(x) - 1}{1-\cos(x)}$$

Solution.

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For which α does the following limit exist?

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Solution. As $x \to x_0 = 0$, we have

- $\frac{1+y}{1-y} = 1 + 2y + o(y)$ and $\frac{1+x^2}{1-x^2} = 1 + 2x^2 + o(x^2)$

For which α does the following limit exist?

$$\lim_{x \to 0} \frac{\left(\frac{1+x^2}{1-x^2}\right) - \alpha \sin(x) - 1}{1 - \cos(x)}$$

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- $\frac{1+y}{1-y} = 1 + 2y + o(y)$ and $\frac{1+x^2}{1-x^2} = 1 + 2x^2 + o(x^2)$
- $\sin x = x + o(x^2)$

For which α does the following limit exist?

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Solution. As $x \to x_0 = 0$, we have

- $\frac{1+y}{1-y} = 1 + 2y + o(y)$ and $\frac{1+x^2}{1-x^2} = 1 + 2x^2 + o(x^2)$
- $\sin x = x + o(x^2)$
- $1 - \cos x = \frac{x^2}{2} + o(x^2)$

For which α does the following limit exist?

$$\lim_{x \to 0} \frac{\left(\frac{1+x^2}{1-x^2}\right) - \alpha \sin(x) - 1}{1 - \cos(x)}$$

Solution. As $x \to x_0 = 0$, we have

- $\frac{1+y}{1-y} = 1 + 2y + o(y)$ and $\frac{1+x^2}{1-x^2} = 1 + 2x^2 + o(x^2)$
- $\sin x = x + o(x^2)$
- $1 - \cos x = \frac{x^2}{2} + o(x^2)$

Then it holds, as $x \to 0$,

$$\frac{\left(\frac{1+x^2}{1-x^2}\right) - \alpha \sin(x) - 1}{1 - \cos(x)} = \frac{1 + 2x^2 - \alpha x - 1 + o(x^2)}{\frac{x^2}{2} + o(x^2)}$$

For which α does the following limit exist?

$$\lim_{x \to 0} \frac{\left(\frac{1+x^2}{1-x^2}\right) - \alpha \sin(x) - 1}{1 - \cos(x)}$$

Solution. As $x \to x_0 = 0$, we have

- $\frac{1+y}{1-y} = 1 + 2y + o(y)$ and $\frac{1+x^2}{1-x^2} = 1 + 2x^2 + o(x^2)$
- $\sin x = x + o(x^2)$
- $1 - \cos x = \frac{x^2}{2} + o(x^2)$

Then it holds, as $x \to 0$,

$$\frac{\left(\frac{1+x^2}{1-x^2}\right) - \alpha \sin(x) - 1}{1 - \cos(x)} = \frac{1 + 2x^2 - \alpha x - 1 + o(x^2)}{\frac{x^2}{2} + o(x^2)}$$

hence $\lim_{x\to 0} \lim_{x\to 0} \frac{\left(\frac{1+x^2}{1-x^2}\right) - \alpha \sin(x) - 1}{1-\cos(x)}$ exists if and only if $\alpha = 0$, and in that case, the limit is 4.