Mathematical Analysis I: Lecture 31

Lecturer: Yoh Tanimoto

12/11/2020

Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 10:00 from 17 November on.

- Today: Apostol Vol. 1, Chapter 7.4-5.

For f n-times differentiable, the following holds (as we prove later). With the convention $f^{(0)}(x) = f(x)$,

$$
\begin{aligned}
f(x) &= f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) + \\
&\quad \cdots + \frac{1}{n!}(x - x_0)^n f^{(n)}(x_0) + o((x - x_0)^n) \\
&= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + o((x - x_0)^n)
\end{aligned}
$$

The part $\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$ is called the Taylor polynomial of f.

Lemma

Let f, g differentiable n times in (a,b) and $x_0 \in (a,b)$. Suppose that $g^{(k)}(x) \neq 0$ for $x \neq x_0, 0 \leq k \leq n$ but $f^{(k)}(x_0) = g^{(k)}(x_0) = 0$ for $0 \leq k \leq n-1$. Then for any $x \neq x_0, x \in (a,b)$ there is ξ between x, x_0 such that $\frac{f(x)}{g(x)} = \frac{f^{(n)}(\xi)}{g^{(n)}(\xi)}$.

Proof.

By Cauchy's mean value theorem,

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f'(\xi_1)}{g'(\xi_1)} = \frac{f'(\xi_1)-f'(x_0)}{g'(\xi_1)-g'(x_0)} = \frac{f^{(2)}(\xi_2)}{g^{(2)}(\xi_2)} = \cdots = \frac{f^{(n)}(\xi_n)}{g^{(n)}(\xi_n)}$$

and we put $\xi = \xi_n$. □

Lemma

Let F differentiable n times in $x_0 \in (a,b)$. Then, $F(x) = o((x-x_0)^n)$ as $x \to x_0$ if and only if $F^{(k)}(x_0) = 0$ for $0 \le k \le n$.

Proof.

We know this for $n = 0$ by definition. Let us prove the general case by induction, by assuming that it is true for n.
Let $F(x) = o((x-x_0)^{n+1})$. Then $F^{(k)}(x_0) = 0$ for $0 \le k \le n$ by the hypothesis of induction. The assumption is $0 = \lim_{x\to x_0} \frac{F(x)}{(x-x_0)^{n+1}}$. On the other hand, $\frac{F(x)}{(x-x_0)^n} = \frac{F^{(n)}(\xi)}{n!(\xi - x_0)}$ for some ξ between x, x_0 by the previous Lemma. If $x \to x_0$, $\xi \to x_0$, that is, $0 = \lim_{x\to x_0} \frac{F(x)}{(x-x_0)^{n+1}} = \lim_{\xi\to x_0} \frac{F^{(n)}(\xi)}{n!(\xi-x_0)} = \lim_{\xi\to x_0} \frac{F^{(n)}(\xi) - F^{(n)}(x_0)}{n!(\xi-x_0)} = \frac{F^{(n+1)}(x_0)}{n!}$, hence $F^{(n+1)}(x_0) = 0$.

Proof.

Let $F^{(k)}(x_0) = 0$ for $0 \le k \le n+1$. Then by the Bernoulli-de l'Hôpital theorem,

$$
\begin{aligned}
0 &= \frac{F^{(n+1)}(x_0)}{(n+1)!} = \lim_{x \to x_0} \frac{F^{(n)}(x) - F^{(n)}(x_0)}{(n+1)!(x - x_0)} \\
&= \lim_{x \to x_0} \frac{F^{(n)}(x)}{(n+1)!(x - x_0)} = \lim_{x \to x_0} \frac{F^{(n-1)}(x) - F^{(n-1)}(x_0)}{\frac{(n+1)!}{2}(x - x_0)^2} \\
&= \lim_{x \to x_0} \frac{F^{(n-1)}(x)}{\frac{(n+1)!}{2}(x - x_0)^2} = \lim_{x \to x_0} \frac{F^{(n-2)}(x) - F^{(n-2)}(x_0)}{\frac{(n+1)!}{3!}(x - x_0)^3} \\
&\cdots = \lim_{x \to x_0} \frac{F(x)}{(x - x_0)^{n+1}}.
\end{aligned}
$$

That is, $F(x) = o((x - x_0)^{n+1})$. □

Corollary

Let $f(x)$ differentiable n times at $x_0 \in (a, b)$. Then with

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k,$$

we have $f(x) = P_n(x) + o((x - x_0)^n)$.

Proof.

$D^k(f(x_0) - P_n(x_0)) = 0$ for $0 \leq k \leq n$. By Lemma 2,
$f(x_0) = P_n(x_0) + o((x - x_0)^n)$. □

Example

- $f(x) = e^x$. As $f^{(n)}(x) = e^x$, we have $f^{(n)}(0) = 1$ and hence $e^x = \sum_{k=0}^{n} \frac{x^n}{n!} + o(x^n)$ as $x \to x_0 = 0$. That is, $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + o(x^n)$.
- $f(x) = \sin x$. As $f^{(4n)}(x) = \sin x$, $f^{(4n+1)}(x) = \cos x$, $f^{(4n+2)}(x) = -\sin x$, $f^{(4n+3)}(x) = -\cos x$, we have $f^{(4n)}(0) = 0$, $f^{(4n+1)}(x) = 1$, $f^{(4n+2)}(x) = 0$, $f^{(4n+3)}(x) = -1$, and hence $\sin x = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + o(x^{2n+1})$. That is, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + o(x^{2n+1})$ as $x \to x_0 = 0$.

Very often, the Taylor series converges to the original function $f(x)$, that is,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

holds for some functions ($e^x, \sin x, \cos x, \log(1 + x)$) and for some x. One example we can show easily such convergence is $f(x) = \frac{1}{1-x}$. Indeed, $f'(x) = \frac{1}{(1-x)^2}, f^{(n)}(x) = \frac{n!}{(1-x)^n}$. And hence the Taylor series around $x = 0$ is

$$\sum_{k=0}^{n} \frac{n!x^n}{n!} = \sum_{k=0}^{n} x^n,$$

and we know that this partial sum is $\frac{1-x^n}{1-x}$, which converges to $\frac{1}{1-x}$ for $|x| < 1$. But the series does not converge for if $|x| \geq 0$.

There are functions whose Taylor series converges but not to the original function. For example, if we take

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

then $f^{(n)}(0) = 0$ for all n, hence the Taylor polynomial is identically 0, but the original function $f(x)$ is not identically 0.
The question of for which function the Taylor series converges to the origial function will be studied in Mathematical Analysis II.

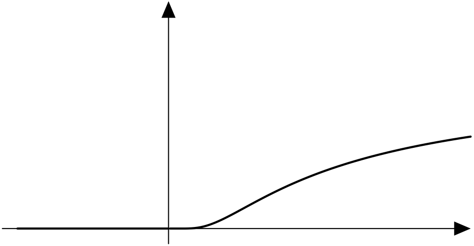

Figure: The graph of a function whose Taylor series converges but not to itself.

Applications to certain limits

Taylor's formula can be used to compute certain indefinite limits.

$$\lim_{x \to 0} \frac{e^x - 1 - x}{\sin(x^2)}$$

As $x \to x_0 = 0$, we have

- $e^x =$

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$$\frac{e^x - 1 - x}{\sin(x^2)} = \frac{\frac{x^2}{2} + o(x^2)}{x^2 + o(x^2)}$$

hence $\lim_{x \to 0} \frac{e^x - 1 - x}{\sin(x^2)} = \frac{1}{2}$.

Applications to certain limits

$$\lim_{x \to 0} \frac{x - \ln(1-x) - 2x\sqrt{1+x}}{\sin(x) - xe^x}$$

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- $\sin x = x + 0 \cdot x^2 + o(x^2)$
- $xe^x = x(1 + x + o(x)) = x + x^2 + o(x^2)$

Then it holds, as $x \to 0$,

$$\frac{x - \ln(1-x) - 2x\sqrt{1+x}}{\sin(x) - xe^x} = \frac{-\frac{1}{2}x^2 + o(x^2)}{-x^2 + o(x^2)}$$

hence $\lim_{x\to 0} \frac{x-\ln(1-x)-2x\sqrt{1+x}}{\sin(x)-xe^x} = \frac{1}{2}$.

Exercises

- Find the n-th order Taylor formula. $f(x) = \cos(x)$ as $x \to 0$.
- Find the n-th order Taylor formula. $f(x) = \log(1 + x)$ as $x \to 0$.
- Find the n-th order Taylor formula. $f(x) = \sin(x^2)$ as $x \to 0$.
- Compute the limit.

$$\lim_{x \to 0} \frac{e^x + \cos(x) - \sin(x) - 2}{\tan(2x^3)}.$$

- For which α does the following limit exist?

$$\lim_{x \to 0} \frac{\ln\left(\frac{1+x^2}{1-x^2}\right) - \alpha \sin(x)}{1 - \cos(x)}$$