Mathematical Analysis I: Lecture 30

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Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 10:00 from 17 November on.
- Today: Apostol Vol. 1, Chapter 7.1-4.

Landau's symbols

Definition

Let I be an open interval, $f, f_1, f_2, g : I \to \mathbb{R}$, $x_0 \in I$ and suppose that $g(x) \neq 0$ in an neighbourhood of $x_0, x \neq x_0$. We write:

- $f(x) = O(g(x))$ (as $x \to x_0$) if there is $M > 0$ such that $|f(x)| \leq M|g(x)|$ in an neighbourhood of x_0.
- $f(x) = o(g(x))$ (as $x \to x_0$) if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$.
- $f_1(x) = f_2(x) + O(g(x))$ ($f_1(x) = f_2(x) + o(g(x))$, respectively) if $f_1(x) - f_2(x) = O(g(x))$ ($= o(g(x))$, respectively).

Landau's symbols

Definition

Similarly, let $f, g : (a, \infty) \to \mathbb{R}$, and suppose that $g(x) \neq 0$ for sufficiently large x (that is, there is $X > 0$ such that $g(x) \neq 0$ if $x > X$). We write:

- $f(x) = O(g(x))$ (as $x \to \infty$) if there is $M > 0$ such that $|f(x)| \leq M|g(x)|$ for sufficiently large x.
- $f(x) = o(g(x))$ (as $x \to \infty$) if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

The cases for $(-\infty, a)$, or $f(x) \to 0$ and the cases in I but $f(x) \to 0$ are analogous.

Landau's symbols

Example

- If $n > 1$, $x^n = o(x)$ as $x \to 0$ (because $\lim_{x\to 0} \frac{x^n}{x} \to 0$).
- $x^n = o(x^m)$ as $x \to 0$ if $n > m$ (because $\lim_{x\to 0} \frac{x^n}{x^m} \to 0$).
- $x^m = o(x^n)$ as $x \to \infty$ se $n > m$ (because $\lim_{x\to\infty} \frac{x^m}{x^n} \to 0$).
- $\log x = o(x)$ as $x \to \infty$ (because $\lim_{x\to\infty} \frac{\log x}{x} \to 0$).
- $\log x = o(\frac{1}{x})$ as $x \to 0$ (because $\lim_{x\to 0} x \log x \to 0$).
- $\sin x = O(x)$ as $x \to 0$ (because $\lim_{x\to 0} \frac{\sin x}{x} \to 1$).
- $\sin x = o(x)$ as $x \to \infty$ (because $\lim_{x\to\infty} \frac{\sin x}{x} \to 0$).
- $\cos x = O(1)$ as $x \to 0$ (because $\lim_{x\to 0} \cos x \to 1$).
- $e^x - 1 = O(x)$ as $x \to 0$ (because $\lim_{x\to 0} \frac{e^x - 1}{x} \to 1$).

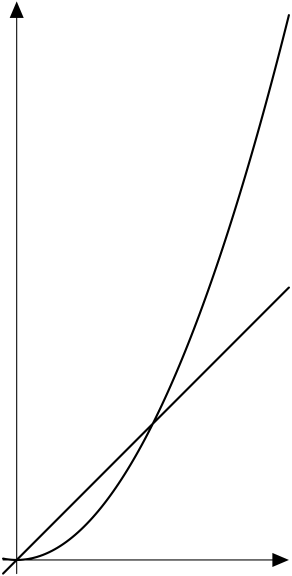

Figure: Landau's symbol. $x^2 = o(x)$ as $x \to 0$, but $x = o(x^2)$ as $x \to \infty$.

Landau's symbols

Lemma

Let us consider the behaviour $x \to x_0 = 0$ (other cases are analogous).

(a) *Let $a, b \in \mathbb{R}$. If $f(x) = O(h(x)), g(x) = O(h(x))$, then $af(x) + bg(x) = O(h(x))$.*

(b) *Let $a, b \in \mathbb{R}$. If $f(x) = o(h(x)), g(x) = o(h(x))$, then $af(x) + bg(x) = o(h(x))$.*

(c) *If $g(x) = o(h(x))$, then $f(x)g(x) = o(f(x)h(x))$ (Similarly, if $g(x) = O(h(x))$, then $f(x)g(x) = O(f(x)h(x))$).*

(d) *If $f(x) = o(h(x))$, then $f(x) = O(h(x))$.*

(e) *Let $f(x) = o(h(x))$ and $f(0) = 0, \lim_{x\to 0} g(x) = 0$. Then $f(g(x)) = o(h(g(x)))$. (Similarly if $f(x) = O(h(x))$, then $f(g(x)) = O(h(g(x)))$.)*

Proof.

1. (a) We have $|f(x)| \leq M_1|x|, |g(x)| \leq M_2|h(x)|$, hence $|af(x) + bg(x)| \leq |a||f(x)| + |b||g(x)| \leq (|a|M_1 + |b|M_2)|h(x)|$.
2. (b) Analogous.
3. (c) If $\lim_{x\to 0} \frac{g(x)}{h(x)} = 0$, then $\lim_{x\to 0} \frac{f(x)g(x)}{f(x)h(x)} = \lim_{x\to 0} \frac{g(x)}{h(x)} = 0$.
4. (d) If $\lim_{x\to 0} \frac{f(x)}{h(x)} \to 0$, then $\left|\frac{f(x)}{h(x)}\right| < M$ for x close enough to 0, hence $|f(x)| < M|h(x)|$.
5. (e) Let us define

$$u(k) = \begin{cases} \frac{f(k)}{h(k)} & \text{if } k \neq 0 \\ 0 & \text{if } k = 0. \end{cases}$$

Then $u(k)$ is continuous at $k = 0$ because $\frac{f(k)}{h(k)} \to 0$ as $k \to 0$. We have $f(g(x)) = h(g(x))u(g(x))$, and Altogether, $\lim_{x\to 0} \left|\frac{f(g(x))}{h(g(x))}\right| = \lim_{x\to 0} \left|\frac{h(g(x))u(g(x))}{h(g(x))}\right| = 1 \cdot 0 = 0$. The other claim is analogous.

□

Landau's symbols

Example

As $x \to 0$,

- $\sin(x^2) = O(x^2)$, because $\sin(y) = O(y)$ and we put $y = x^2$.
- $e^{x^3} - 1 = O(x^3)$, because $e^y - 1 = O(y)$, and we put $y = x^3$.
- $\sin^2(x) = O(x^2)$, because $\sin x = O(x)$, and hence $\sin^2 x = O(x \sin x) = O(x^2)$.

Second order Taylor Formula

We have defined derivative by $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$. If f is differentiable at x_0, then we have $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$, or equivalently,

$$\lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0} - \frac{f'(x_0)(x-x_0)}{x-x_0}\right)$$
$$= \lim_{x\to x_0}\frac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{x-x_0} = 0,$$

therefore, $f(x)-f(x_0)-f'(x_0)(x-x_0) = o(x-x_0)$. This means that we can approximate f **to the first order** by $f(x_0)+f'(x_0)(x-x_0)$. This is indeed called the first order Taylor formula.

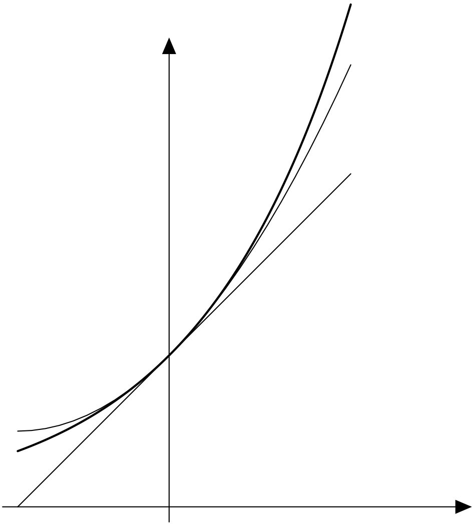

Figure: The second order Taylor formula. We approximate a general function by a second order polynomial.

Second order Taylor Formula

The Taylor formula can be extended to higher order.

Theorem (Second order Taylor formula)

Let f differentiable in (a, b) and twice differentiable at $x_0 \in (a, b)$. Then $f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) + o((x - x_0)^2)$ as $x \to x_0$.

Proof.

Let us put $P_2(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0)$. Then $P_2'(x) = f'(x_0) + (x - x_0)f''(x_0)$. Furthermore, the first order Talylor formula holds for f': $f'(x) = f'(x_0) + (x - x_0)f''(x_0) + o(x - x_0)$ as $x \to x_0$. That is,

$$\lim_{x \to x_0} \frac{D(f(x) - P_2(x))}{D((x - x_0)^2)}$$
$$= \lim_{x \to x_0} \frac{f'(x) - f'(x_0) - (x - x_0)f''(x_0)}{2(x - x_0)} = \frac{1}{2}(f''(x_0) - f''(x_0)) = 0.$$

By the Bernoulli-de l'Hôpital theorem,

$$\lim_{x \to x_0} \frac{f(x) - P_2(x)}{(x - x_0)^2} = 0.$$

that is, $f(x) = P_2(x) + o((x - x_0)^2)$. □

Second order Taylor Formula

Example

As $x \to 0$,

- $e^x = 1 + x + \frac{x^2}{2} + o(x^2)$
- $\log(1 + x) = x - \frac{x^2}{2} + o(x^2)$.
- $\sin(x) = x + o(x^2)$.
- $\cos(x) = 1 - \frac{x^2}{2} + o(x^2)$.

Exercises

- Find the second order Taylor formula. $f(x) = \sin(x^2)$ as $x \to 0$.
- Find the second order Taylor formula. $f(x) = \sqrt{x^2}$ as $x \to 1$.
- Find the second order Taylor formula. $f(x) = \sin(x) - 1$ as $x \to \frac{\pi}{2}$.
- Find the second order Taylor formula. $f(x) = \frac{e^x - 1}{\cos x}$ as $x \to 0$.