Mathematical Analysis I: Lecture 29

Lecturer: Yoh Tanimoto

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Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 15:00-16:30 until 10th November. Then move to Tuesday morning.
- Today: Apostol Vol. 1, Chapter 7.14.

Bernoulli-de l'Hôpital rule

Let us recall the mean value theorem of Cauchy: let f, g be continuous in $[a, b]$ and differentiable in (a, b). Then there is $y \in (a, b)$ such that

$$f'(y)(g(b) - g(a)) = g'(y)(f(b) - f(a)).$$

(Bernoulli-)de l'Hôpital rule is a useful tool to compute limits of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Bernoulli-de l'Hôpital rule

Theorem (Bernoulli-de l'Hôpital, case 1)

Let $a < x_0$, f, g differentiable in (a, x_0) such that $g'(x) \neq 0$ for x sufficiently close to $x_0, x \neq x_0$, $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^-} g(x) = 0$, $\lim_{x \to x_0^-} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$. Then $g(x) \neq 0$ for x close to $x_0, x \neq x_0$ and $\lim_{x \to x_0^-} \frac{f(x)}{g(x)} = L$.

A similar result holds for right limits.

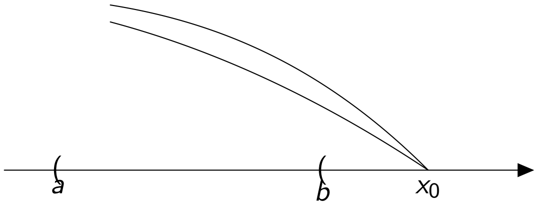

Figure: Theorem of de l'Hôpital. The limit $\lim_{x \to x_0^-} \frac{f(x)}{g(x)}$ is determined by $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$.

Proof.

We can extend f, g to $(a, x_0]$ by putting $f(x_0) = g(x_0) = 0$, such that they are continuous. By the hypothesis we may assume that $g'(x) \neq 0$ in (b, x_0). Let $x \in (b, x_0)$, by Lagrange's mean value theorem, there is $y \in (x, x_0)$ such that $g(x) = g(x) - g(x_0) = g'(y)(x - x_0) \neq 0$, in particular, $g(x) \neq 0$.

By Cauchy's mean value theorem, for x above, there is $y \in (x, x_0)$ such that $f'(y)(g(x) - g(x_0)) = g'(y)(f(x) - f(x_0))$, that is,

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(y)}{g'(y)}.$$

If $x \to x_0$, such y tends to x_0. Because $\lim_{y \to x_0^-} \frac{f'(y)}{g'(y)} = L$ by the hypothesis it holds that $\lim_{x \to x_0^-} \frac{f(x)}{g(y)} = L$. □

Bernoulli-de l'Hôpital rule

Example

- Consider $\frac{e^x-1}{\sin(2x)}$. The limit $x \to 0$ is of the form $\frac{0}{0}$. It holds that $(\sin(2x))' = 2\cos(2x) \neq 0$ as $x \to 0$. In addition $(e^x - 1)' = e^x$. Therefore, $\lim_{x\to 0} \frac{e^x-1}{\sin x} = \lim_{x\to 0} \frac{e^x}{2\cos(2x)} = \frac{1}{2}$.
- $\lim_{x\to 0} \frac{x}{e^x-1}$

Bernoulli-de l'Hôpital rule

Example

- Consider $\frac{e^x-1}{\sin(2x)}$. The limit $x \to 0$ is of the form $\frac{0}{0}$. It holds that $(\sin(2x))' = 2\cos(2x) \neq 0$ as $x \to 0$. In addition $(e^x - 1)' = e^x$. Therefore, $\lim_{x\to 0} \frac{e^x-1}{\sin x} = \lim_{x\to 0} \frac{e^x}{2\cos(2x)} = \frac{1}{2}$.
- $\lim_{x\to 0} \frac{x}{e^x-1} = \frac{1}{e^0} = 1$.
- $\lim_{x\to 0} \frac{x^2}{\cos x - 1}$

Bernoulli-de l'Hôpital rule

Example

- Consider $\frac{e^x-1}{\sin(2x)}$. The limit $x \to 0$ is of the form $\frac{0}{0}$. It holds that $(\sin(2x))' = 2\cos(2x) \neq 0$ as $x \to 0$. In addition $(e^x - 1)' = e^x$. Therefore, $\lim_{x\to 0} \frac{e^x-1}{\sin x} = \lim_{x\to 0} \frac{e^x}{2\cos(2x)} = \frac{1}{2}$.
- $\lim_{x\to 0} \frac{x}{e^x-1} = \frac{1}{e^0} = 1$.
- $\lim_{x\to 0} \frac{x^2}{\cos x-1} = \lim_{x\to 0} \frac{2x}{-\sin x} = \frac{2}{-\cos 0} = -2$.

Bernoulli-de l'Hôpital rule

Theorem (Bernoulli-de l'Hôpital, case 2)

Let f, g differentiable in (a, ∞) such that $g'(x) \neq 0$ for x sufficiently large, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$, $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$. Then $g(x) \neq 0$ for x sufficiently large and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$.

Proof.

Let $F(x) = f(\frac{1}{x}), G(x) = g(\frac{1}{x})$. Note that, as $x \to \infty$, we have $\frac{1}{x} \to 0^+$, and $F'(x) = -\frac{1}{x^2} f'(\frac{1}{x}), G'(x) = -\frac{1}{x^2} g'(\frac{1}{x})$. Then for sufficiently small x, $G'(x) \neq 0$ because $g'(\frac{1}{x}) \neq 0$ for such x. By applying case 1, we obtain

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{F(x)}{G(x)} = \lim_{x\to 0^+} \frac{F'(x)}{G'(x)} = \lim_{x\to 0^+} \frac{-x^2 f'(\frac{1}{x})}{-x^2 g'(\frac{1}{x})} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

as desired. □

Bernoulli-de l'Hôpital rule

Example

- $\lim_{x\to\infty} \frac{\sin(\frac{1}{x^2})}{\frac{1}{x^2}}$

Bernoulli-de l'Hôpital rule

Example

- $\lim_{x\to\infty} \frac{\sin(\frac{1}{x^2})}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{-\frac{2}{x^3}\cos(\frac{1}{x^2})}{-\frac{2}{x^3}} = 1.$

Bernoulli-de l'Hôpital rule

Theorem (Bernoulli-de l'Hôpital, case 3)

Let $a < x_0$, f, g *differentiable in* (a, x_0) *such that* $g'(x) \neq 0$ *for* x *sufficiently close to* x_0, $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = +\infty$, $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$. *Then* $g(x) \neq 0$ *for* x *sufficiently close to* x_0 *and* $\lim_{x \to x_0} \frac{f(x)}{g(x)} = L$.

Proof.

Let $\varepsilon > 0$. By the hypothesis, there is b such that $\left|\frac{f'(y)}{g'(y)} - L\right| < \frac{\varepsilon}{3}$ for $y \in (b, x_0)$. In addition, there is $\tilde{b}$ such that $b < \tilde{b} < x_0$ and in $(\tilde{b}, x_0)$ $f(x) > 2f(b) > 0, g(x) > 2g(b) > 0$. Then the function $h(x) = \frac{1-\frac{g(b)}{g(x)}}{1-\frac{f(b)}{f(x)}}$ is continuous on $(\tilde{b}, x_0]$ and its value at x_0 is 1 . Furthermore, it holds that

$$\frac{f(x) - f(b)}{g(x) - g(b)} \cdot h(x) = \frac{f(x) - f(b)}{g(x) - g(b)} \cdot \frac{1 - \frac{g(b)}{g(x)}}{1 - \frac{f(b)}{f(x)}} = \frac{f(x)}{g(x)}.$$

Proof.

Let $\tilde{\tilde{b}}$ such that $|h(x) - 1| < \frac{\varepsilon}{3L+1}$ for $x \in (\tilde{\tilde{b}}, x_0)$. By Cauchy's mean value theorem, there is $y \in (b, x)$ such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(b)}{g(x) - g(b)} \cdot h(x) = \frac{f'(y)}{g'(y)} h(x).$$

Now $\left|\frac{f(x)}{g(x)} - L\right| = \left|\frac{f'(y)}{g'(y)} h(x) - L\right| < \left|\frac{f'(y)}{g'(y)} - L\right| (1 + \frac{\varepsilon}{3L+1}) + L|h(x) - 1| < \frac{\varepsilon}{3} + \frac{\varepsilon^2}{3} + \frac{\varepsilon}{3} < \varepsilon$. □

Bernoulli-de l'Hôpital rule

Theorem (Bernoulli-de l'Hôpital, case 4)

Let f, g *be differentiable* (a, ∞) *such that* $g'(x) \neq 0$ *as* $x \to \infty$, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = +\infty$, $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$. *Then* $g(x) \neq 0$ *for* x *sufficiently large and* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$.

Proof.

Consider $F(y) = f(\frac{1}{y}), G(y) = g(\frac{1}{y})$. Since $\frac{1}{y} \to \infty$ as $y \to 0^+$, and $D(F(y)) = \frac{Df(\frac{1}{y})}{-y^2}, D(G(y)) = \frac{Dg(\frac{1}{y})}{-y^2}$ we can apply case 3 and $L = \lim_{y\to 0^+} \frac{DF(y)}{DG(y)} = \lim_{y\to 0^+} \frac{F(y)}{G(y)} = \lim_{x\to\infty} \frac{f(x)}{g(x)}$. $\square$

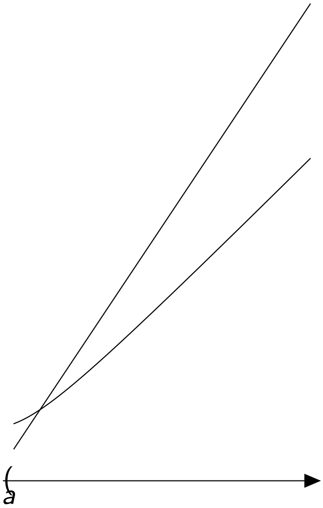

Figure: Theorem of de l'Hôpital. The limit $\lim_{x\to\infty}\frac{f(x)}{g(x)}$ is determined by $\lim_{x\to\infty}\frac{f'(x)}{g'(x)}$.

Example

- Let us compute $\lim_{x\to\infty} \frac{x^2}{e^x}$. If the limit $\lim_{x\to\infty} \frac{2x}{e^x}$ exists, then by the de l'Hôpital rule, they should coincide. The latter exists if $\lim_{x\to\infty} \frac{2}{e^x}$ exists, and it does: it is 0. Therefore, the second limit exists and it is 0, and hence the first limit exists and it is 0.
- $\lim_{x\to 0} \frac{\sin 2x}{\sin x}$

Bernoulli-de l'Hôpital rule

Example

- Let us compute $\lim_{x\to\infty} \frac{x^2}{e^x}$. If the limit $\lim_{x\to\infty} \frac{2x}{e^x}$ exists, then by the de l'Hôpital rule, they should coincide. The latter exists if $\lim_{x\to\infty} \frac{2}{e^x}$ exists, and it does: it is 0. Therefore, the second limit exists and it is 0, and hence the first limit exists and it is 0.
- $\lim_{x\to 0} \frac{\sin 2x}{\sin x} = \lim_{x\to 0} \frac{2\cos 2x}{\cos x} = 2.$
- $\lim_{x\to 0} \frac{\log x}{1/\tan x}$

Example

- Let us compute $\lim_{x\to\infty} \frac{x^2}{e^x}$. If the limit $\lim_{x\to\infty} \frac{2x}{e^x}$ exists, then by the de l'Hôpital rule, they should coincide. The latter exists if $\lim_{x\to\infty} \frac{2}{e^x}$ exists, and it does: it is 0. Therefore, the second limit exists and it is 0, and hence the first limit exists and it is 0.
- $\lim_{x\to 0} \frac{\sin 2x}{\sin x} = \lim_{x\to 0} \frac{2\cos 2x}{\cos x} = 2$.
- $\lim_{x\to 0} \frac{\log x}{1/\tan x} = \lim_{x\to 0} \frac{1/x}{1/\sin^2 x} = 0$.
- $\lim_{x\to 0} \frac{\log(\sin x)}{\log x}$

Example

- Let us compute $\lim_{x\to\infty}\frac{x^2}{e^x}$. If the limit $\lim_{x\to\infty}\frac{2x}{e^x}$ exists, then by the de l'Hôpital rule, they should coincide. The latter exists if $\lim_{x\to\infty}\frac{2}{e^x}$ exists, and it does: it is 0. Therefore, the second limit exists and it is 0, and hence the first limit exists and it is 0.
- $\lim_{x\to 0}\frac{\sin 2x}{\sin x} = \lim_{x\to 0}\frac{2\cos 2x}{\cos x} = 2$.
- $\lim_{x\to 0}\frac{\log x}{1/\tan x} = \lim_{x\to 0}\frac{1/x}{1/\sin^2 x} = 0$.
- $\lim_{x\to 0}\frac{\log(\sin x)}{\log x} = \lim_{x\to 0}\frac{\frac{\cos x}{\sin x}}{x} = 1$.
- $\lim_{x\to\infty}\frac{x^n}{e^x}$

Example

- Let us compute $\lim_{x\to\infty}\frac{x^2}{e^x}$. If the limit $\lim_{x\to\infty}\frac{2x}{e^x}$ exists, then by the de l'Hôpital rule, they should coincide. The latter exists if $\lim_{x\to\infty}\frac{2}{e^x}$ exists, and it does: it is 0. Therefore, the second limit exists and it is 0, and hence the first limit exists and it is 0.
- $\lim_{x\to 0}\frac{\sin 2x}{\sin x} = \lim_{x\to 0}\frac{2\cos 2x}{\cos x} = 2$.
- $\lim_{x\to 0}\frac{\log x}{1/\tan x} = \lim_{x\to 0}\frac{1/x}{1/\sin^2 x} = 0$.
- $\lim_{x\to 0}\frac{\log(\sin x)}{\log x} = \lim_{x\to 0}\frac{\frac{\cos x}{\sin x}}{x} = 1$.
- $\lim_{x\to\infty}\frac{x^n}{e^x} = 0$.
- $\lim_{x\to\infty}\frac{\log\cosh x}{x}$

Example

- Let us compute $\lim_{x\to\infty} \frac{x^2}{e^x}$. If the limit $\lim_{x\to\infty} \frac{2x}{e^x}$ exists, then by the de l'Hôpital rule, they should coincide. The latter exists if $\lim_{x\to\infty} \frac{2}{e^x}$ exists, and it does: it is 0. Therefore, the second limit exists and it is 0, and hence the first limit exists and it is 0.
- $\lim_{x\to 0} \frac{\sin 2x}{\sin x} = \lim_{x\to 0} \frac{2\cos 2x}{\cos x} = 2$.
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- $\lim_{x\to 0} \frac{\log(\sin x)}{\log x} = \lim_{x\to 0} \frac{\frac{\cos x}{\sin x}}{x} = 1$.
- $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$.
- $\lim_{x\to\infty} \frac{\log\cosh x}{x} = \lim_{x\to\infty} \frac{\sinh x/\cosh x}{1} = 1$.

Exercises

- Compute the limit. $\lim_{x \to 0} \frac{\sin^2 x}{x^2}$.
- Compute the limit. $\lim_{x \to 0} \frac{\sin x - x}{x^3}$.
- Compute the limit. $\lim_{x \to \infty} \frac{\log x^3 + 1}{\log x}$.
- Compute the limit. $\lim_{x \to 0} \frac{x}{\log x}$.