Mathematical Analysis I: Lecture 28

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Exercises

Determine where the function is increasing or decreasing. $f(x) = x^3 - 3x$.
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Hence f is increasing when $f'(x) > 0$, that is $x < -1$ or $x > 1$ and f is decreasing when $f'(x) < 0$, that is $-1 < x < 1$.

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$f'(x) = e^x - 1$.
Hence f is increasing when $f'(x) > 0$, that is $x > 0$ and f is decreasing when $f'(x) < 0$, that is $x < 0$.

Determine where the function is increasing or decreasing. $f(x) = x + \frac{1}{x}$, $x > 0$.

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$f'(x) = 1 - \frac{1}{x^2}$.

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$f'(x) = 1 - \frac{1}{x^2}$.
Hence f is increasing when $f'(x) > 0$, that is $x > 1$ and f is decreasing when $f'(x) < 0$, that is $x < 1$.

Determine where the function is increasing or decreasing. $f(x) = \frac{x}{x^2+1}$.
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$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{-(x-1)(x+1)}{(x^2+1)^2}$.

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$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{-(x-1)(x+1)}{(x^2+1)^2}$.
Hence f is increasing when $f'(x) > 0$, that is $-1 < x < 1$ and f is decreasing when $f'(x) < 0$, that is $x < -1, x > 1$.

Find the local maxima and minima using the second derivative.
$f(x) = 2x^3 - 3x^2$.
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Solution.
$f'(x) = 6x^2 - 6x = 6x(x-1)$.
Hence $f'(x) = 0$ if and only if $x = 0, 1$. $f''(x) = 12x - 6$, $f''(0) = -6 < 0$ hence $x = 0, f(0) = 0$ is a local maximum, while $f''(1) = 6 > 0$ hence $x = 1, f(1) = -1$ is a local minimum.

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$f(x) = xe^x$.
Solution.
$f'(x) = e^x + xe^x = e^x(x+1)$.
Hence $f'(x) = 0$ if and only if $x = -1$. $f''(x) = 2e^x + xe^x$,
$f''(-1) = \frac{2}{e} - \frac{1}{e} = \frac{1}{e} > 0$ hence $x = -1, f(-1) = -\frac{1}{e}$ is a local minimum.

Find the asymptotes of $f(x) = \sqrt{x^2 + 1}$.
Solution.

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Solution.

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x\to\infty} \sqrt{1+\frac{1}{x^2}} = 1$$

and

$$\lim_{x\to\infty} \sqrt{x^2+1} - x = \lim_{x\to\infty} \frac{(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x} = 0.$$

Hence $y = x$ is an asymptote for $x \to \infty$. Similarly,

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x\to-\infty} -\sqrt{1+\frac{1}{x^2}} = -1$$

and

$$\lim_{x\to-\infty} \sqrt{x^2+1} - (-x) = \lim_{x\to-\infty} \frac{1}{\sqrt{x^2+1}-x} = 0.$$

Hence $y = -x$ is an asymptote for $x \to -\infty$.

Find the asymptotes of $f(x) = x + \frac{1}{x}$.
Solution.

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Solution.

$$\lim_{x\to\infty} \frac{x + \frac{1}{x}}{x} = \lim_{x\to\infty} 1 + \frac{1}{x^2} = 1,$$

and

$$\lim_{x\to\infty} x + \frac{1}{x} - x = \lim_{x\to\infty} \frac{1}{x} = 0.$$

hence $y = x$ is an asymptote for $x \to \infty$. Similarly, $y = -x$ an asymptote for $x \to -\infty$.

Sketch the graph of $f(x) = \frac{x^2-5}{x-3}$.

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- Domain: $x \neq 3$.
- Vertical asymptote: $x = 3$.
  Oblique asymptotes: $\lim_{x\to\infty} \frac{x^2-5}{x(x-3)} = 1$, and

$$\lim_{x\to\infty} \frac{x^2-5}{x-3} - x = \lim_{x\to\infty} \frac{x^2-5-x(x-3)}{x-3} = \lim_{x\to\infty} \frac{3x-5}{x-3} = 3,$$

  hence $y = x + 3$ is an asymptote for $x \to \infty$. Similarly, $y = x + 3$ is an asymptote for $x \to -\infty$.

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- Domain: $x \neq 3$.
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  hence $y = x + 3$ is an asymptote for $x \to \infty$. Similarly, $y = x + 3$ is an asymptote for $x \to -\infty$.
- $f'(x) = \frac{2x(x-3)-(x^2-5)}{(x-3)^2} = \frac{x^2-6x+5}{(x-3)^2} = \frac{(x-1)(x-5)}{(x-3)^2}$. $f'(x) > 0$ if and only if $x < 1, x > 5$ and $f'(x) < 0$ if and only if $1 < x < 3, 3 < x < 5$.

Sketch the graph of $f(x) = \frac{x^2-5}{x-3}$.
Solution.

- Domain: $x \neq 3$.
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  hence $y = x+3$ is an asymptote for $x \to \infty$. Similarly, $y = x+3$ is an asymptote for $x \to -\infty$.
- $f'(x) = \frac{2x(x-3)-(x^2-5)}{(x-3)^2} = \frac{x^2-6x+5}{(x-3)^2} = \frac{(x-1)(x-5)}{(x-3)^2}$. $f'(x) > 0$ if and only if $x < 1, x > 5$ and $f'(x) < 0$ if and only if $1 < x < 3, 3 < x < 5$.
- From this, $x = 1$ is a local maximum and $x = 5$ is a local minimum.

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- Domain: $x \neq 3$.
- Vertical asymptote: $x = 3$.
  Oblique asymptotes: $\lim_{x\to\infty} \frac{x^2-5}{x(x-3)} = 1$, and

$$\lim_{x\to\infty} \frac{x^2-5}{x-3} - x = \lim_{x\to\infty} \frac{x^2-5-x(x-3)}{x-3} = \lim_{x\to\infty} \frac{3x-5}{x-3} = 3,$$

  hence $y = x+3$ is an asymptote for $x \to \infty$. Similarly, $y = x+3$ is an asymptote for $x \to -\infty$.
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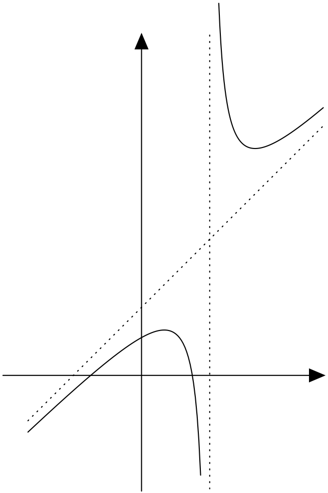
x		1		3		5	
$f'(x)$	$+$	0	$-$	nd	$-$	0	$+$
$f''(x)$		$-$		nd		$+$	
$f(x)$	$\nearrow$	2	$\searrow$	nd	$\searrow$	10	$\nearrow$



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- Domain: $x \neq 1$, and $\frac{x^3}{x-1} \geq 0$, that is $x > 1$ or $x \leq 0$.
- Vertical asymptote:

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  Oblique asymptotes: $\lim_{x\to\infty} \sqrt{\frac{x^3}{x-1}}\frac{1}{x} = \lim_{x\to\infty} \sqrt{\frac{x^3}{x^2(x-1)}} = 1$, and

$$\lim_{x\to\infty} \sqrt{\frac{x^3}{x-1}} - x = \lim_{x\to\infty} x\left(\sqrt{\frac{x}{x-1}} - 1\right)$$

$$= \lim_{x\to\infty} x\frac{(\sqrt{x} - \sqrt{x-1})(\sqrt{x} + \sqrt{x-1})}{\sqrt{x-1}(\sqrt{x} + \sqrt{x-1})}$$

$$= \lim_{x\to\infty} x\frac{x-(x-1)}{\sqrt{x-1}(\sqrt{x} + \sqrt{x-1})} = \frac{1}{2}$$

  hence $y = x + \frac{1}{2}$ is an asymptote for $x \to \infty$.

- Similarly, $\lim_{x\to-\infty}\sqrt{\frac{x^3}{x-1}}\frac{1}{x} = \lim_{x\to-\infty} -\sqrt{\frac{x^3}{x^2(x-1)}} = -1$, and

$$\begin{aligned}
\lim_{x\to-\infty}\sqrt{\frac{x^3}{x-1}} - (-x) &= \lim_{x\to-\infty} x\left(-\sqrt{\frac{-x}{-x+1}}+1\right) \\
&= \lim_{x\to-\infty} x\frac{(\sqrt{-x}-\sqrt{-x+1})(\sqrt{-x}+\sqrt{-x+1})}{\sqrt{-x+1}(\sqrt{-x}+\sqrt{-x+1})} \\
&= \lim_{x\to-\infty} x\frac{-x-(-x+1)}{\sqrt{-x+1}(\sqrt{-x}+\sqrt{-x+1})} = -\frac{1}{2}
\end{aligned}$$

hence $y = -x - \frac{1}{2}$ is an asymptote for $x \to -\infty$.

- Similarly, $\lim_{x\to-\infty}\sqrt{\frac{x^3}{x-1}}\frac{1}{x} = \lim_{x\to-\infty} -\sqrt{\frac{x^3}{x^2(x-1)}} = -1$, and

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hence $y = -x - \frac{1}{2}$ is an asymptote for $x \to -\infty$.

- $f'(x) = \frac{3x^2(x-1)-x^3}{2(x-1)^2 f(x)} = \frac{x^2(2x-3)}{2(x-1)^2 f(x)}$. $f'(x) > 0$ if $x > \frac{3}{2}$ and $f'(x) < 0$ if $x < \frac{3}{2}$. $f(\frac{3}{2}) = \sqrt{\frac{27}{4}}$.

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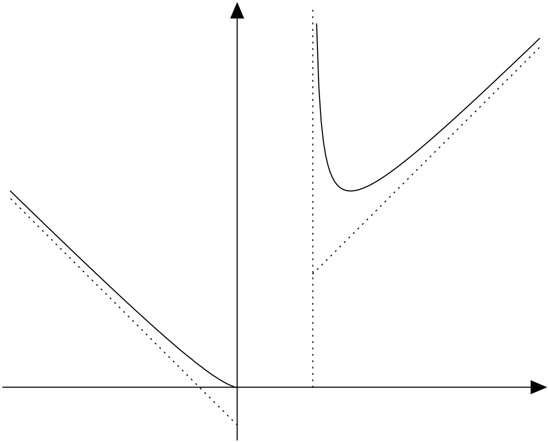
- $f'(x) = \frac{3x^2(x-1)-x^3}{2(x-1)^2 f(x)} = \frac{x^2(2x-3)}{2(x-1)^2 f(x)}$. $f'(x) > 0$ if $x > \frac{3}{2}$ and $f'(x) < 0$ if $x < \frac{3}{2}$. $f(\frac{3}{2}) = \sqrt{\frac{27}{4}}$.

x		0		1		$\frac{3}{2}$	
$f'(x)$	$-$		nd		$-$	0	$+$
$f''(x)$			nd			$+$	
$f(x)$	$\searrow$	0		nd	$\searrow$	$\sqrt{\frac{27}{4}}$	$\nearrow$



A truck is to be driven 300 miles on a freeway at a constant speed of x miles per hour. Speed laws require $30 < x < 60$. Assume that fuel is consumed at the rate of $2 + x^2/600$ gallons per hour. Which speed should the track driver go to save the fuel cost?

Solution.

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Solution.

The truck has to drive fo $300/x$ hours, and then consumes $f(x) = \frac{300}{x}(2 + \frac{x^2}{600}) = \frac{600}{x} + \frac{x}{2}$ gallons.

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Solution.
The truck has to drive fo $300/x$ hours, and then consumes $f(x) = \frac{300}{x}(2 + \frac{x^2}{600}) = \frac{600}{x} + \frac{x}{2}$ gallons.
Considering this as a function of x, we find its minimum in $30 < x < 60$.
$f'(x) = -\frac{600}{x^2} + \frac{1}{2}$, hence $f(x) = 0$ if $x = \sqrt{1200} \cong 34.6$. $f''(x) = \frac{1200}{x^3}$, hence this is a local minimum.