Mathematical Analysis I: Lecture 27

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Annoucements

- Today: Apostol Vol. 1, Chapter 4.18-20.

Symmetry of functions

Recall that a function is a subset in $\mathbb{R} \times \mathbb{R}$ in the sense that it collects all the points $\{(x, y) \in \mathbb{R} \times \mathbb{R} : y = f(x)\}$. This is the graph itself.
We can consider certain operations on a function.

- **Translation**. If $g(x) = f(x - a) + b$ for some function f, g, then the graph of g is obtained by translating the graph of f by (a, b). Indeed, if (x, y) is on the graph of f, then $(x + a, y + b)$ is on the graph of g.
- **Reflection**. If $g(x) = f(-x)$ for some function f, g, then the graph of g is obtained by reflecting the graph of f with respect to $x = 0$. Indeed, if (x, y) is on the graph of f, then $(-x, y)$ is on the graph of g.
- If $g(x) = f(-(x - 2a))$ for some function f, g, then the graph of g is obtained by reflecting the graph of f with respect to $x = a$.
- **Scaling**. If $g(x) = bf(x/a)$ for some function f, g and $a, b > 0$, then the graph of g is obtained by scaling the graph of f by a in the x-direction and b in the y-direction. Indeed, if (x, y) is on the graph of f, then (ax, by) is on the graph of g.

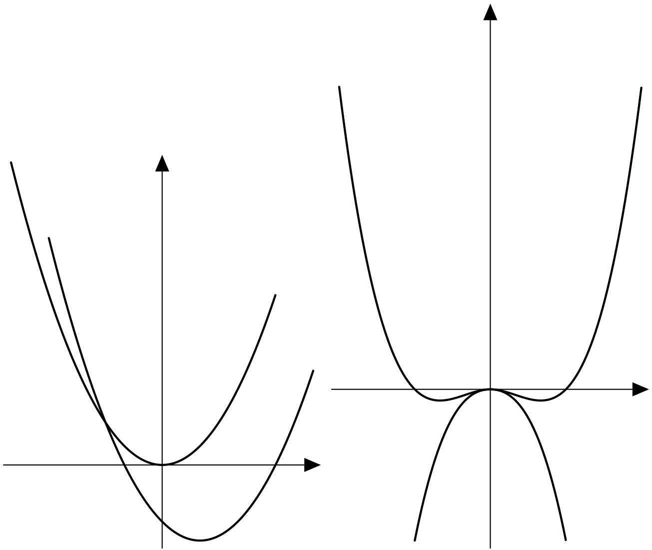

Figure: Left:the graphs of x^2 and $(x-\frac{1}{2})^2-1$. Right:the graphs of x^3-x^2 and $-x^3-x^2$.

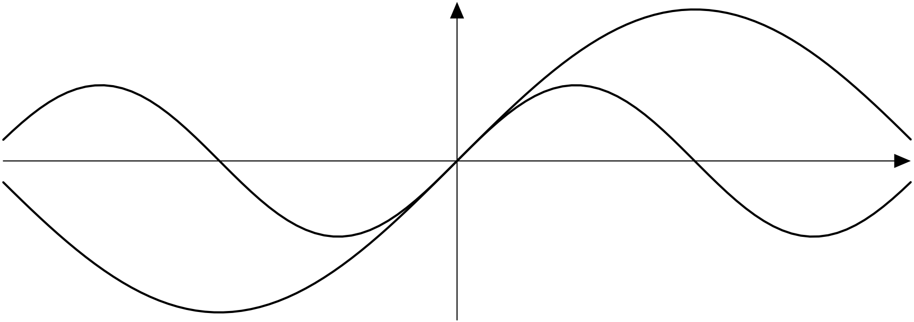

Figure: The graphs of $\sin x$ and $2\sin(x/2)$.

Symmetry of functions

A graph or a function may have a **symmetry**. A function f is said to have a symmetry if it is invariant under certain operations.

- **Translation symmetry**. If $f(x) = f(x-a)$, then the graph of f remains invariant under the translation $(a, 0)$.
- **Reflection**. If $f(x) = f(-x)$, then the graph of f is invariant under the reflection respect to $x = 0$ and f is said to be even.
- $f(x) = -f(-x)$, f is said to be **odd**.
- $f(x) = f(-(x-2a))$ has reflection symmetry with respect to $x = a$.

Example

- The graph of $\sin x$ is invariant under 2π translation and under the reflection with respect to $\frac{\pi}{2}$, because $\sin(x + 2\pi) = \sin(x)$ and $\sin(-(x-\pi)) = -\sin(x-\pi) = \sin(x)$. On the other hand, $\sin(-x) = -\sin x$, hence $\sin x$ is an odd function.
- If $f(x) = (x - \frac{1}{2})^2 - 1$ is invariant under the reflection with respect to $x = \frac{1}{2}$ because $((-(x-1)) - \frac{1}{2})^2 - 1 = (-x + \frac{1}{2})^2 - 1 = (x - \frac{1}{2})^2 - 1$.

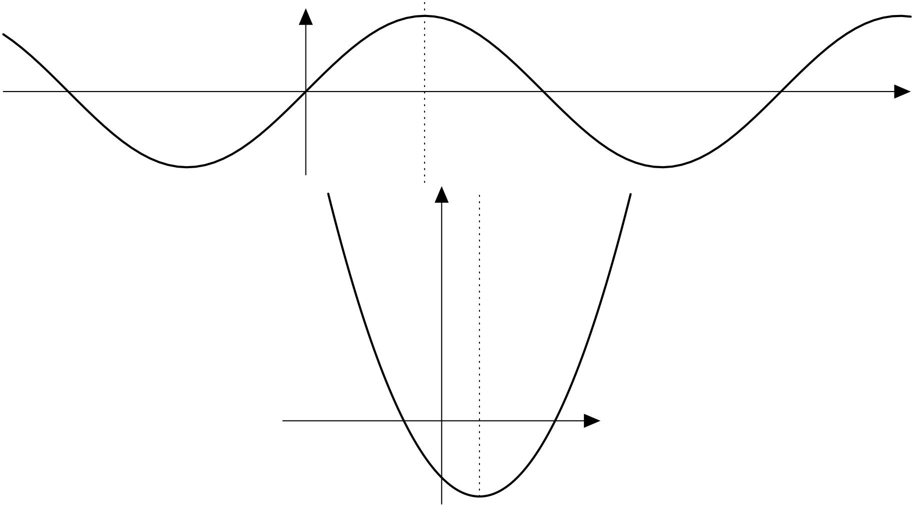

Figure: $\sin x$ is invariant under 2π translation and under the reflection with respect to $\frac{\pi}{2}$. $(x - \frac{1}{2})^2 + 1$ is invariant under the reflection with respect to $x = \frac{1}{2}$.

Curve sketching

The graph of a function f can be **qualitatively** drawn as follows.

(0) Determine the (natural) domain A of definition of f.

(0.5) Se if f has a symmetry or a period.

(1) Study the sign of f: where $f(x) > 0, = 0, < 0$ hold.

(2) Determine the asymptotes.

(3) Study the sign of f' and find stationary points (where $f'(x) = 0$).

(4) Study the stationary points and find local minima and maxima (either by the second derivative or the first).

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(3) $f'(x) = -4(2x-1)e^{-(2x-1)^2}$. $f'(x) = 0 \Leftrightarrow 2x - 1 = 0 \Leftrightarrow x = \frac{1}{2}$.
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(4) $f''(x) = (16(2x-1)^2 - 8)e^{-(2x-1)^2} = (64x^2 - 64x + 8)e^{-(2x-1)^2}$.

x		$\frac{1}{2}$	
$f'(x)$	$+$	0	$-$
$f''(x)$		$-$	
$f(x)$	$\nearrow$	1	$\searrow$

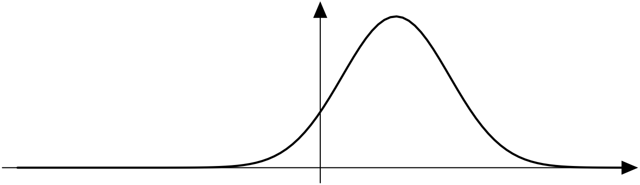

Figure: The graph of $f(x) = e^{-(2x-1)^2}$.

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(2) The domain is $(0, \pi)$, so we need to check $\{0, \pi\}$. $\lim_{x \to 0} f(x) = \lim_{x \to \pi} f(x) = \infty$. The asymptotes are $x = 0, \pi$.

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(3) $f'(x) = -\frac{\cos x}{\sin x}$. $f'(x) = 0 \Leftrightarrow \cos x = 0 \Leftrightarrow x = \frac{\pi}{2}$. $f(\frac{\pi}{2}) = 0$. $f'(x) < 0$ if $x \in (0, \frac{\pi}{2})$, and $f'(x) > 0$ if $x \in (\frac{\pi}{2}, \pi)$.

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(4) $f''(x) = \frac{1}{\sin^2 x} > 0$.

x		$\frac{\pi}{2}$	
$f'(x)$	$-$	0	$+$
$f''(x)$		$+$	
$f(x)$	$\searrow$	0	$\nearrow$

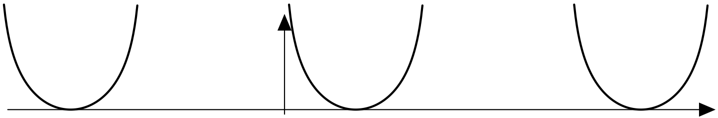

Figure: The graph of $f(x) = \log(\frac{1}{\sin x})$.

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We can draw the graphs of $f(x) = 1 - x^2$ and $g(x) = e^x - 1$, and prove that there are two solutions of the equation $f(x) = g(x)$.

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We can draw the graphs of $f(x) = 1 - x^2$ and $g(x) = e^x - 1$, and prove that there are two solutions of the equation $f(x) = g(x)$.
Indeed, let us consider the function $h(x) = g(x) - f(x) = e^x + x^2 - 2$ and it suffices to find all x such that $h(x) = g(x) - f(x) = 0$. We have $\lim_{x\to\pm\infty} g(x) - f(x) = \infty$ and $g(0) - f(0) = (1-1) - 1 = -1$. By the intermediate value theorem, there are solutions in $x > 0$ e $x < 0$.
Moreover, $h'(x) = e^x + 2x$, hence there is only one stationary point (because in $x > 0$ $h'(x)$ is positive and it is negative for sufficiently small x, while $g''(x) - f''(x) = e^x + 2$ is positive, therefore, $g'(x) - f'(x)$ is monotonically increasing). Therefore, $h(x) = g(x) - f(x)$ is decreasing in a negative half line and is increasing in the rest, hence it can have only two points x where $h(x) = 0$.

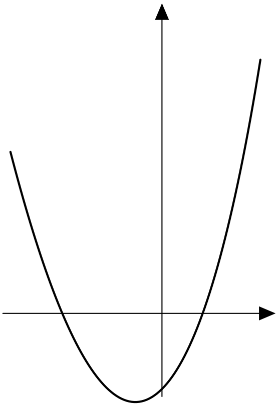

Figure: The graph of $h(x) = e^x + x^2 - 2$. It crosses the x-axis twice and only twice.

Some applications of the minimum/maximum finding

If one can express a problem as a problem of finding the maximum or the minimum of a function, we can solve it using derivatives and graphs.

- Among all rectangles of given perimeter $2r$, which one has the largest area?

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- Among all rectangles of given perimeter $2r$, which one has the largest area?

  Let the vertical side x, then $0 \leq x \leq r$ and the other side is $r - x$, hence the area is $x(r-x)$. We need to find the maximum of $f(x) = x(r-x)$ on the domain $\{x : 0 < x < r\}$. We have $\lim_{x\to 0} f(x) = \lim_{x \to r} f(x) = 0$, while $f'(x) = r - 2x$, and hence there is a stationary point at $x = \frac{r}{2}$, and $f''(x) = -2$, hence this is a local maximum. There is no other stationary points, and $f(0) = f(r) = 0$, hence this is the maximum.

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- The geometric mean $\sqrt{ab}$ is smaller than or equal to the arithmetic mean $\frac{a+b}{2}$.

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- The geometric mean $\sqrt{ab}$ is smaller than or equal to the arithmetic mean $\frac{a+b}{2}$.

  Let us fix $P = \sqrt{ab}$ and put $a = x$, then $b = \frac{P^2}{x}$ and $0 < x$. Let us find the minimum of $f(x) = \frac{x+\frac{P^2}{x}}{2}$. This tends to ∞ as $x \to 0$ or $x \to \infty$. On the other hand, $f'(x) = \frac{1}{2}(1 - \frac{P^2}{x^2})$, and hence there is only one stationary point at $x = P$, and $f''(x) = \frac{2P^2}{x^3}$, hence this is a local minimum, and is the minimum. At $x = P$, we have $f(P) = P$. Hence we have $P \leq \frac{x+\frac{P^2}{x}}{2}$.

Exercises

- Sketch the graph of $f(x) = \frac{x^2-4}{x-3}$.
- Sketch the graph of $f(x) = \sqrt{\frac{x^3}{x-1}}$.
- A truck is to be driven 300 miles on a freeway at a constant speed of x miles per hour. Speed laws require $30 < x < 60$. Assume that fuel is consumed at the rate of $2 + x^2/600$ gallons per hour. Which speed should the track driver go to save the fuel cost?