Mathematical Analysis I: Lecture 26

Lecturer: Yoh Tanimoto

04/11/2020

Start recording...

Annoucements

- Today: Apostol Vol. 1, Chapter 4.17-18.

Higher derivatives

As we saw before, if f is defined on an open interval and is differentiable on each point of I, then f' defines a new function on I, the (first) derivative. It may happen that f' is again differentiable on each point of I, and it defines a further new function f'', the **second derivative**. If f'' is again differentiable, one can also define the third derivative, and so on. We denote the n-th derivative by $f^{(n)}$, or $D^n f$, $\frac{d^n f}{dx^n}$.

Higher derivatives

Example

- If $f(x) = x^4$, then

Higher derivatives

Example

- If $f(x) = x^4$, then $f'(x) = 4x^3$,

Higher derivatives

Example

- If $f(x) = x^4$, then $f'(x) = 4x^3$, $f''(x) = 12x^2$,

Higher derivatives

Example

- If $f(x) = x^4$, then $f'(x) = 4x^3$, $f''(x) = 12x^2$, $f^{(3)}(x) = 24x$, and so on. This f is infinitely many times differentiable.
- If $f(x) = \sin x$, then

Higher derivatives

Example

- If $f(x) = x^4$, then $f'(x) = 4x^3$, $f''(x) = 12x^2$, $f^{(3)}(x) = 24x$, and so on. This f is infinitely many times differentiable.
- If $f(x) = \sin x$, then $f'(x) = \cos x$,

Higher derivatives

Example

- If $f(x) = x^4$, then $f'(x) = 4x^3$, $f''(x) = 12x^2$, $f^{(3)}(x) = 24x$, and so on. This f is infinitely many times differentiable.
- If $f(x) = \sin x$, then $f'(x) = \cos x$, $f''(x) = -\sin x$,

Higher derivatives

Example

- If $f(x) = x^4$, then $f'(x) = 4x^3$, $f''(x) = 12x^2$, $f^{(3)}(x) = 24x$, and so on. This f is infinitely many times differentiable.
- If $f(x) = \sin x$, then $f'(x) = \cos x$, $f''(x) = -\sin x$, $f^{(3)}(x) = -\cos x$,

Higher derivatives

Example

- If $f(x) = x^4$, then $f'(x) = 4x^3$, $f''(x) = 12x^2$, $f^{(3)}(x) = 24x$, and so on. This f is infinitely many times differentiable.
- If $f(x) = \sin x$, then $f'(x) = \cos x$, $f''(x) = -\sin x$, $f^{(3)}(x) = -\cos x$, $f^{(4)}(x) = \sin x$ and so on. Again this is infinitely many times differentiable.
- Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + \frac{x}{2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$, then

$$f'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \frac{x^2}{x^2}\cos\left(\frac{1}{x}\right) + \frac{1}{2} & \\ = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) + \frac{1}{2} & \text{for } x \neq 0 \\ \frac{1}{2} & \text{for } x = 0 \end{cases}$$

and this is not continuous. In particular, f is only once differentiable.

Higher derivatives

The second derivative is useful to study whether the stationary point (or a **critical point**) is a maximum or a minimum, and also to study the shape of the graph.

Lemma

Suppose that f is differentiable in an open interval I and at x_0 it is twice differentiable.

- *If x_0 is a stationary point and $f''(x_0) > 0$ ($f''(x_0) < 0$, respectively), then f takes a local minimum (a local maximum, respectively) at x_0 .*
- *If x_0 is a local minimum (a local maximum, respectively), then $f''(x_0) \geq 0$ ($f''(x_0) \leq 0$, respectively).*

Proof.

- Let $f''(x_0) > 0$. Then there is $\epsilon > 0$ such that $\frac{f'(x_0+h)-f'(x_0)}{h} = \frac{f'(x_0+h)}{h} > 0$ for $|h| < \epsilon$. This means that $f'(x_0 + h) > 0$ for $h > 0$ and $f'(x_0 + h) < 0$ for $h < 0$, and hence f is monotonically decreasing in $(x_0 - \epsilon, x_0)$ and increasing in $(x_0, x_0 + \epsilon)$, that is, f takes a minimum at x_0.
- If x_0 a local minimum and suppose that $f''(x_0) < 0$, then x_0 would be a local maximum and it would contradict the previous point.
- Other cases are analogous.

□

Example

- Let $f(x) = x^2$. We have $f'(x) = 2x$ and $x = 0$ is a stationary point. As $f''(x) = 2$, x is a minimum.
- Let $f'(x) = x^3 - 3x$. We have $f'(x) = 3x^2 - 3$ and $x = 1, -1$ are stationary points. As $f''(x) = 6x$, f takes a maximum at $x = -1$ and a minimum at $x = 1$.

Note that, for $a, b \in \mathbb{R}$ and $t \in [0,1]$. Then $ta + (1-t)b$ is a point between a, b. Indeed, if $a < b$, then $a = ta + (1-t)a < ta + (1-t)b < tb + (1-t)b = b$ (the case $b < a$ is analogous).

Definition

Let f be defined on an interval I. We say that f is convex (concave, respectively) if for any $a, b \in I$ and $t \in [0,1]$ it holds that

$$f(ta + (1-t)b) \le tf(a) + (1-t)f(b)$$
$$(\text{ respectively } f(ta + (1-t)b) \ge tf(a) + (1-t)f(b)).$$

Note that $(ta + (1-t)b, tf(a) + (1-t)f(b))$ defines a segment between $(a, f(a))$ and $(b, f(b))$. Indeed, the slope from the point $(a, f(a))$ to such a point is $\frac{(1-t)(f(b)-f(a))}{(1-t)(b-a)} = \frac{f(b)-f(a)}{b-a}$, which does not depend on t.

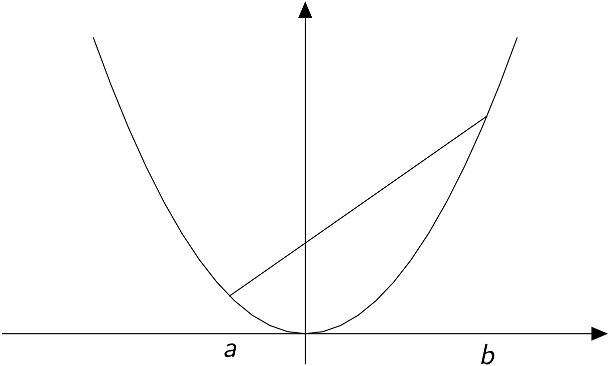

Figure: A convex function. The graph is below the segment between any pair of points $(a, f(a)), (b, f(b))$.

Convexity and concavity

Theorem

Assume that f is continuous on $[a, b]$, differentiable on (a, b). If f' is monotonically nondecreasing (nonincreasing, respectively), then f is convex (concave, respectively). In particular, if $f''(x) > 0$ ($f''(x) < 0$, respectively) for $x \in (a, b)$, then f is convex (concave, respectively).

Under certain conditions, it can also be shown that f is convex, then $f'' > 0$. We omit the proof.

Example

- Let $f(x) = x^2$. As $f''(x) = 2$, f is convex.
- Let $f'(x) = x^3$. As $f''(x) = 6x$, f is concave on $(-\infty, 0)$ and convex on $(0, \infty)$.

Proof.

Let $x < y$ in $[a, b]$ and $t \in (0, 1)$. Let $z = tx + (1-t)y$. We have to prove that $f(z) \leq tf(x) + (1-t)f(y)$, or equivalently, $t(f(z) - f(x)) \leq (1-t)(f(y) - f(z))$.

By the mean value theorem, there are points c, d such that $x < c < z$ and $z < d < y$ such that $f(z) - f(x) = f'(c)(z - x)$ and $f(y) - f(z) = f'(d)(y - z)$. As f' is nondecreasing, $f'(c) \leq f'(d)$ and hence, using $t(z - x) = (1-t)(y - z)$,

$$\begin{aligned} t(f(z) - f(x)) &= tf'(c)(z - x) \\ &\leq f'(d)t(z - x) = f'(d)(1-t)(y - z) = (1-t)(f(y) - f(z)). \end{aligned}$$

□

Asymptotes

The graph of some function may approach a straight line. A more precise concept of this is asymptotes.

Definition

- Let f be defeind on (a, ∞). If $\lim_{a \to \infty} f(x) = L$, then we say that $y = L$ is a **horizontal asymptote** (analogous for $-\infty$).
- Let f be defined on (a, b). If $\lim_{x \to a^+} |f(x)| \to \infty$, then $x = a$ is called a **vertical asymptote**. (analogous for b).
- Let f be defined on (a, ∞). If there is $A, B \in \mathbb{R}$ such that $\lim_{x \to \infty} \frac{f(x)}{x} = A$ and $\lim_{x \to \infty} f(x) - Ax = B$, then we say that $y = Ax + B$ is an **oblique asymptote** (analogous for $-\infty$).

Asymptotes

Example

- Let $f(x) = \tanh x$. We know that $\lim_{x \to \infty} \tanh x = 1, \lim_{x \to -\infty} \tanh x = -1$, hence $y = 1, -1$ are the horizontal asymptotes of $\tanh x$.
- Let $f(x) = \frac{1}{x}$ on $(-\infty, 0) \cup (0, \infty)$. We know that $\lim_{x \to 0^+} \frac{1}{x} = \infty$, $\lim_{x \to 0^-} \frac{1}{x} = -\infty$, and hence $x = 0$ is a vertical asymptote of $\frac{1}{x}$. $y = 0$ is a horizontal asymptote of $\frac{1}{x}$ because $\lim_{x \to \pm\infty} \frac{1}{x} = 0$.

Asymptotes

Example

- Let $f(x) = x \tanh x$. Then, we see that $\lim_{x\to\infty} \frac{x \tanh x}{x} = \lim_{x\to\infty} \tanh x = 1$ and

$$\begin{aligned}\lim_{x\to\infty} x \tanh x - x &= \lim_{x\to\infty} x \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} - 1 \right) \\ &= \lim_{x\to\infty} \frac{-2xe^{-x}}{e^x + e^{-x}} = 0,\end{aligned}$$

hence $y = x$ is an oblique asymptote. Similarly, $y = -x$ an oblique asymptote for $x \to -\infty$.

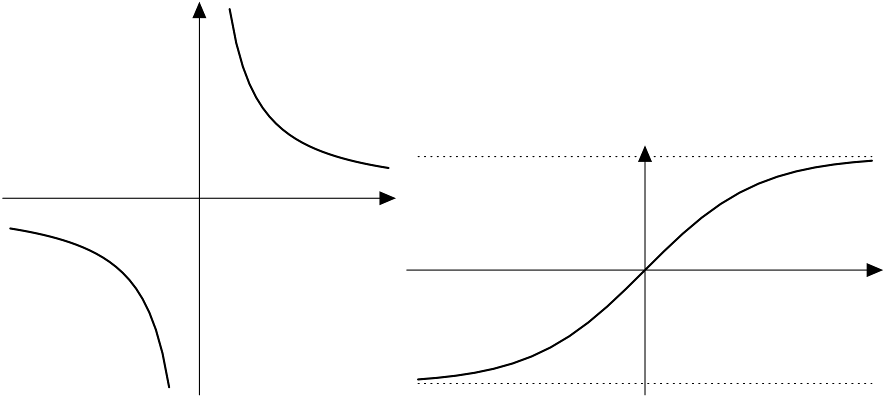

Figure: The asymptotes for $\frac{1}{x}$ and $\tanh x$.

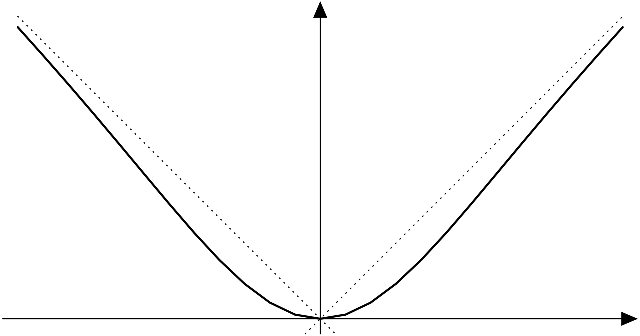

Figure: The oblique asymptotes for $x \tanh x$.

Exercises

- Find the local maxima and minima using the second derivative. $f(x) = 2x^3 - 3x^2$.
- Find the local maxima and minima using the second derivative. $f(x) = xe^x$.
- Find the asymptotes of $f(x) = \sqrt{x^2 + 1}$.
- Find the asymptotes of $f(x) = x + \frac{1}{x}$.