Mathematical Analysis I: Lecture 25

Lecturer: Yoh Tanimoto

02/11/2020

Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 15:00-16:30 until 10th November. Then move to Tuesday morning.
- Today: Apostol Vol. 1, Chapter 4.14-16.

Theorem (Rolle)

Let f be continuous in $[a, b]$ and differentiable in (a, b). If $f(a) = f(b)$, then there is $x_0 \in (a, b)$ such that $f'(x_0) = 0$.

Proof.

If f is constant, then $f'(x) = 0$ for all $x \in (a, b)$.
If f is not constant, then by Theorem of Weierstrass, f has a minimum and a maximum. As f is not constant, one of them must be different from $f(a) = f(b)$. Therefore, we take x_0 that is either minimum or maximum, and $a \neq x_0 \neq b$. Let us take an open interval containing x_0. Now $f'(x_0) = 0$ by Theorem of the last week. $\square$

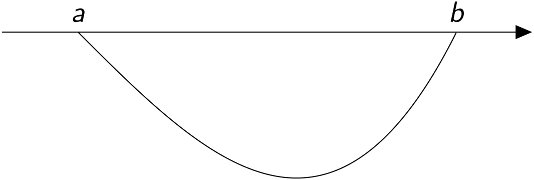

Figure: A non constant function, continuous in $[a, b]$ and differentiable in (a, b), must have a stationary point.

Theorem (Lagrange's mean value theorem)

Let f be continuous in $[a, b]$ and differentiable in (a, b). Then there is $x_0 \in (a, b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(x_0)$.

Proof.

Let $g(x) = f(x) - \frac{(f(b)-f(a))x}{b-a}$, which is continuous in $[a, b]$ and differentiable in (a, b). Then $g(a) = \frac{f(a)b-f(b)a}{b-a} = g(b)$, and by Theorem 1 there is x_0 such that $g'(x_0) = 0$. This implies $f'(x_0) - \frac{f(b)-f(a)}{b-a} = 0$. □

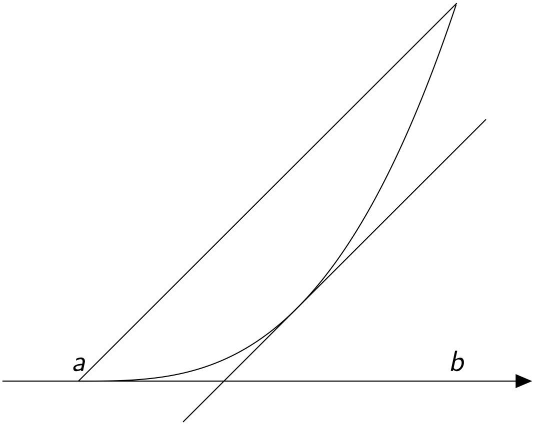

Figure: A function continuous in $[a, b]$ and differentiable in (a, b), must have a point where the deriative is equal to the mean slope.

Applications of derivative

Corollary

Let f be continuous in $[a, b]$ and differentiable in (a, b). If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.

Proof.

Let $x < y \in [a, b]$. By Theorem 2, there is $x_0 \in (x, y)$ such that $\frac{f(y)-f(x)}{y-x} = f'(x_0) = 0$, therefore, $f(x) = f(y)$. $\square$

Applications of derivative

Corollary

Let f be continuous in $[a, b]$ e derivabile in (a, b).

- *If $f'(x) \geq 0$ (> 0, respectively) for all $x \in (a, b)$, then f is monotonically non decreasing (increasing, respectively).*
- *If $f'(x) \leq 0$ (< 0, respectively) for all $x \in (a, b)$, then f is monotonically non increasing (decreasing, respectively).*

Proof.

Let $x < y \in (a, b)$. By Theorem 2, there is $x_0 \in (x, y)$ such that $\frac{f(y)-f(x)}{(y-x)} = f'(x_0)$. If $f'(x_0) \geq 0 (> 0)$, then $f(y) - f(x) \geq 0 (> 0)$, that is f is monotonically non decreasing (increasing, respectively).
The case $f'(x) \leq 0 (< 0)$ is analogous. $\square$

Applications of derivative

Example

- $f(x) = x^2$.

Applications of derivative

Example

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Applications of derivative

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Example

- $f(x) = x^2$. $f'(x) = 2x$, hence f is decreasing if $x < 0$, $x = 0$ is the only one stationary point, and is increasing if $x > 0$.
- $f(x) = \sin x$. $f'(x) = \cos x$, hence f is increasing if $x \in (-\frac{\pi}{2} + 2\pi n, \frac{\pi}{2} + 2\pi n)$ for $n \in \mathbb{Z}$, $x = \frac{\pi}{2} + 2\pi n, -\frac{\pi}{2} + 2\pi n$ are stationary points, and f is decreasing if $x \in (\frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n)$.

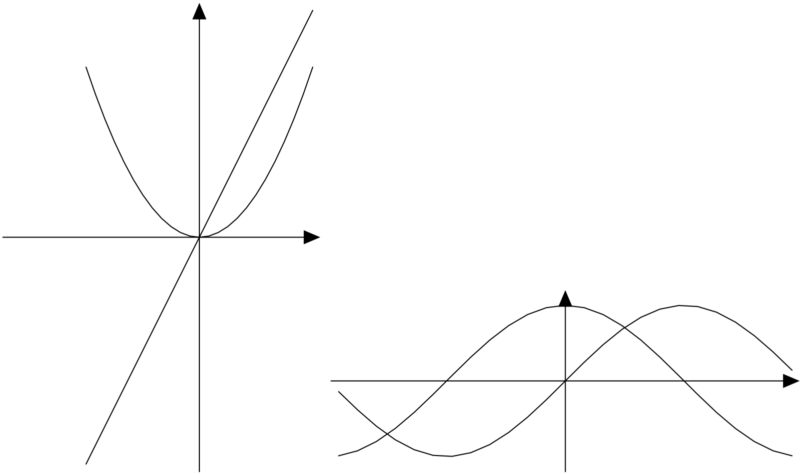

Figure: A function and its derivative. When the derivative is positive (negative) in an interval, the function is increasing (decreasing).

Applications of derivative

Theorem

Let f be continuous in $[a, b]$ and differentiable in (a, b). Let $c \in (a, b)$.

- *If $f'(x) > 0$ for $x \in (a, c)$ and $f'(x) < 0$ for $x \in (c, b)$, then f has a maximum at c.*
- *If $f'(x) < 0$ for $x \in (a, c)$ and $f'(x) > 0$ for $x \in (c, b)$, then f has a minimum at c.*

Proof.

If $f'(x) > 0$ for $x \in (a, c)$, then it is increasing there and continuous at c, therfore, for any $x \in (a, b)$ it holds that $f(c) \geq f(x)$. On the other hand, as $f'(x) < 0$ for $x \in (c, b)$, and $f(c) \geq f(x)$ for $x \in (c, b)$.
The second case is analogous. $\square$

Applications of derivative

Example

- $f(x) = x^3 - x$.

Applications of derivative

Example

- $f(x) = x^3 - x$. $f'(x) = 3x^2 - 1$,

Example

- $f(x) = x^3 - x$. $f'(x) = 3x^2 - 1$, and $f'(x) > 0$ if and only if $x < -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} < x$, and $f'(x) < 0$ if and only if $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$.

Applications of derivative

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Example

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- $f(x) = \cosh x$.

Applications of derivative

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Applications of derivative

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- $f(x) = \cosh x$. $f'(x) = \sinh x$, and $f'(x) > 0$ if and only if $x > 0$, and $f'(x) < 0$ if and only if $x < 0$.

Example

- $f(x) = x^3 - x$. $f'(x) = 3x^2 - 1$, and $f'(x) > 0$ if and only if $x < -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} < x$, and $f'(x) < 0$ if and only if $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$. Therefore, x takes a maximum at $x = -\frac{1}{\sqrt{3}}$ and a minimum $x = \frac{1}{\sqrt{3}}$. As f is differentiable in $\mathbb{R}$, there is no other local maximum or minimum.
- $f(x) = \cosh x$. $f'(x) = \sinh x$, and $f'(x) > 0$ if and only if $x > 0$, and $f'(x) < 0$ if and only if $x < 0$. Therefore, x takes a minimum at $x = 0$ and no other minimum or maximum.

Applications of derivative

Example

- $f(x) = x^3 - x$. $f'(x) = 3x^2 - 1$, and $f'(x) > 0$ if and only if $x < -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} < x$, and $f'(x) < 0$ if and only if $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$. Therefore, x takes a maximum at $x = -\frac{1}{\sqrt{3}}$ and a minimum $x = \frac{1}{\sqrt{3}}$. As f is differentiable in $\mathbb{R}$, there is no other local maximum or minimum.
- $f(x) = \cosh x$. $f'(x) = \sinh x$, and $f'(x) > 0$ if and only if $x > 0$, and $f'(x) < 0$ if and only if $x < 0$. Therefore, x takes a minimum at $x = 0$ and no other minimum or maximum.
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Applications of derivative

Example

- $f(x) = x^3 - x$. $f'(x) = 3x^2 - 1$, and $f'(x) > 0$ if and only if $x < -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} < x$, and $f'(x) < 0$ if and only if $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$. Therefore, x takes a maximum at $x = -\frac{1}{\sqrt{3}}$ and a minimum $x = \frac{1}{\sqrt{3}}$. As f is differentiable in $\mathbb{R}$, there is no other local maximum or minimum.
- $f(x) = \cosh x$. $f'(x) = \sinh x$, and $f'(x) > 0$ if and only if $x > 0$, and $f'(x) < 0$ if and only if $x < 0$. Therefore, x takes a minimum at $x = 0$ and no other minimum or maximum.
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Applications of derivative

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Applications of derivative

Example

- $f(x) = x^3 - x$. $f'(x) = 3x^2 - 1$, and $f'(x) > 0$ if and only if $x < -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} < x$, and $f'(x) < 0$ if and only if $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$. Therefore, x takes a maximum at $x = -\frac{1}{\sqrt{3}}$ and a minimum $x = \frac{1}{\sqrt{3}}$. As f is differentiable in $\mathbb{R}$, there is no other local maximum or minimum.
- $f(x) = \cosh x$. $f'(x) = \sinh x$, and $f'(x) > 0$ if and only if $x > 0$, and $f'(x) < 0$ if and only if $x < 0$. Therefore, x takes a minimum at $x = 0$ and no other minimum or maximum.
- $f(x) = \sinh x$. $f'(x) = \cosh x$ and $\cosh x > 0$, and hence $f(x)$ is monotonically increasing.

Note that, even if $f'(x) > 0$ at one point, it does not mean that f is monotonically increasing in a neighbourhood of x. Indeed, a counterexample is given by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + \frac{x}{2} & \text{for } x \neq 0 \\ \frac{1}{2} & \text{for } x = 0 \end{cases}.$$

As we have seen, this function without the part $\frac{x}{2}$ is differentiable, and it has the derivative 0 at $x = 0$. Therefore, with $\frac{x}{2}$, it is still differentiable and $f'(0) = \frac{1}{2} > 0$.
Yet, f is not monotonically increasing in any interval $(-\epsilon, \epsilon)$.

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Applications of derivative

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + \frac{x}{2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}.$$

Yet, f is not monotonically increasing in any interval $(-\epsilon, \epsilon)$. To see this, note that

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \frac{x^2}{x^2}\cos\left(\frac{1}{x}\right) + \frac{1}{2} & \\ = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) + \frac{1}{2} & \text{for } x \neq 0 \\ \frac{1}{2} & \text{for } x = 0 \end{cases}.$$

and for any $\epsilon > 0$, there is $x < \epsilon$ such that $f'(x) < 0$: for example, one can take $x = \frac{1}{2\pi n}$ for sufficiently large n. Then the term $2x \sin \frac{1}{x} = 0$, while $-\cos\frac{1}{x} = -1$, and then $f'(x) = -\frac{1}{2}$.
Note that the derivative $f'(x)$ is discontinuous in this case.

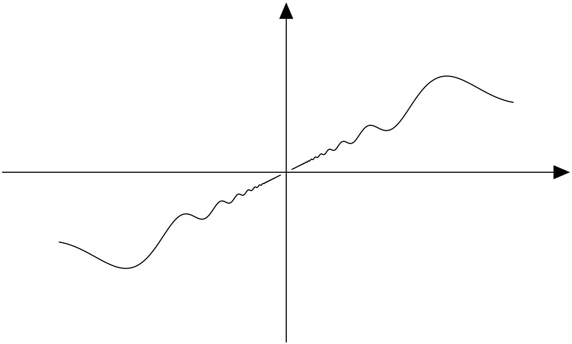

Figure: A function f such that $f'(0) > 0$ but is not monotonically increasing in any interval containing $x = 0$.

Applications of derivative

Theorem (Cauchy's mean value theorem)

Let $a < b$, f, g be continuous in $[a, b]$ and differentiable in (a, b). Then there is $x_0 \in (a, b)$ such that $f'(x_0)(g(b) - g(a)) = g'(x_0)(f(b) - f(a))$.

Proof.

Let $h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$, then $h(x)$ is continuous in $[a, b]$ and differentiable on in (a, b).
$h(a) = f(a)g(b) - f(b)g(a) = h(b)$. By Rolle's theorem 1, there is $x_0 \in (a, b)$ such that
$0 = h'(x_0) = f'(x_0)(g(b) - g(a)) - g(x_0)(f(b) - f(a))$. □

Exercises

- Determine where the function is increasing or decreasing.
  $f(x) = x^3 - 3x$.
- Determine where the function is increasing or decreasing.
  $f(x) = e^x - x$.
- Determine where the function is increasing or decreasing.
  $f(x) = x + \frac{1}{x}$, $x > 0$.
- Determine where the function is increasing or decreasing.
  $f(x) = \frac{x}{x^2+1}$.