Mathematical Analysis I: Lecture 24

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Exercises

Compute the derivative of $f(x) = \begin{cases} x^2 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}.$

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Solution. For $x \neq 0$ we know $f'(x) = \begin{cases} 2x & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$. For $x = 0$, let us compute the left and right derivatives. We compute $\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{0-0}{h} = 0$ and $\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h^2-0}{h} = \lim_{h\to 0^+} h = 0$. So the left and right derivatives coincide, therefore, $f'(0) = 0$.

Tell whether $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$ is continuous and is differentiable at $x = 0$.

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Solution. We have $\frac{f(h)-f(0)}{h} = \frac{h \sin \frac{1}{h} - 0}{h} = \sin \frac{1}{h}$ and this does not have the limit $h \to 0$. But it is continuous at $x = 0$, because $|\sin \frac{1}{x}| \leq 1$, hence $\lim_{x\to 0} |x \sin \frac{1}{x}| \leq \lim_{x \to 0} |x| = 0$.

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Solution. We have
$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h} - 0}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0$, hence it is differentiable and in particular continuous.

Compute the derivative of $f(x) = x^3$ based on the definition.

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Solution.

$$\begin{aligned}\lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2.\end{aligned}$$

Compute the derivative of $f(x) = x^2 + x$ based on the definition.

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Solution.

$$\begin{aligned} &\lim_{h\to 0} \frac{(x+h)^2 + (x+h) - x^2 - x}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + (x+h) - x^2 - x}{h} \\ &= \lim_{h\to 0} (2x + h + 1) = 2x + 1. \end{aligned}$$

Compute the derivative: $f(x) = x^2 - \cos(3x)$.

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Solution. By the chain rule and linearity, it is $2x + 3\sin(3x)$.

Compute the derivative: $f(x) = \sqrt{x^2 + 1}$.

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Solution. By the chain rule with $\sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}$, it is $2x \cdot \frac{1}{2}(x^2+1)^{-\frac{1}{2}} = \frac{x}{\sqrt{x^2+1}}$.

Compute the derivative: $f(x) = \sin(\frac{x+2}{e^x})$.

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Solution. By the chain rule,
$f'(x) = \frac{e^x - e^x(x+2)}{e^{2x}} \cos(\frac{x+2}{e^x}) = -e^{-x}(x+1)\cos(\frac{x+2}{e^x})$.

Compute the derivative: $f(x) = \sin(\cos(x^2))$.

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Solution. By the chain rule,
$f'(x) = D(\cos(x^2))\cos(\cos(x^2)) = -2x\sin(x^2)\cos(\cos(x^2))$.

Compute the derivative: $f(y) = \log y$, using that $\log y$ is the inverse function of e^x.

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Solution. With $y = e^x$, $f'(y) = \frac{1}{e^x} = \frac{1}{y}$.

Compute the derivative: $f(y) = \sqrt{y}$ using that $\sqrt{y}$ is the inverse function of x^2.

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Solution. With $y = x^2$, $f'(y) = \frac{1}{2x} = \frac{1}{2\sqrt{y}}$.

Find the stationary points of $y = x^3 - 3x^2 + 3x$.

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Solution. With $f(x) = x^3 - 3x^2 + 3x$,
$f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x - 1)^2$, hence $x = 1$ is the only stationary point.

Find the stationary points of $y = \sin(x^2)$.

Find the stationary points of $y = \sin(x^2)$.
Solution. With $f(x) = \sin(x^2)$, $f'(x) = 2x\cos(x^2)$, and $f'(x) = 0$ if and only if $x = 0$ or $\cos(x^2) = 0$, hence $x = 0$ or $x = \pm\sqrt{\frac{n\pi}{2}}$ for $n \in \mathbb{N}$ odd.

Consider the relation $y^2 - x^2 = 1$. Write y as an explicit function of x, and take the derivative. Differentiate it implicitly and find a relation.

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Solution. $y(x)^2 = x^2 + 1$ and hence $y(x) = \pm\sqrt{x^2+1}$ and $y'(x) = \pm\frac{x}{\sqrt{x^2+1}}$, we see the relation $y'(x)y(x) = x$.

By differentiating the relation, we obtain $2y(x)y'(x) = 2x$, and hence $y(x)y'(x) = x$.

Consider the relation $y^5 + xy - 2x^3 = 0$. Check that $(x, y) = (1, 1)$ satisfy this equation. Assume that this defines an implicit function $y(x)$, and compute $y'(1)$.

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Solution. $1^5 + 1 \cdot 1 - 2 \cdot 1^3 = 0$. We have $5y'(x)y(x)^4 + y + xy'(x) - 6x^2 = 0$, and hence $y'(1) = \frac{6-1}{5+1} = \frac{6}{5}$.

A boat sails parallel to a straight beach at a constant speed of 12 miles per hour, staying 4 miles offshore. How fast is it approaching a lighthouse on the shoreline at the instant it is exactly 5 miles from the lighthouse?

A boat sails parallel to a straight beach at a constant speed of 12 miles per hour, staying 4 miles offshore. How fast is it approaching a lighthouse on the shoreline at the instant it is exactly 5 miles from the lighthouse?
Solution. Let us say that at time t the boat is at the position $(12t, 4)$, and the lighthouse is at $(0, 0)$. The distance between the lighthouse and the boat is $r(t) = \sqrt{(12t)^2 + 4^2} = 4\sqrt{9t^2 + 1}$, or $r(t)^2 = (12t)^2 + 4^2$.
The speed with which the boat approaches the lighthouse is $r'(t)$. By differentiating the above relation by t, we have $2r(t)r'(t) = 288t$.
Furthermore, When $r(t) = 5$, we have $t = \pm\frac{1}{4}$. Therefore,
$2 \cdot 5r'(\pm\frac{1}{4}) = \pm 72$ and $r'(t) = \pm\frac{36}{5}$.

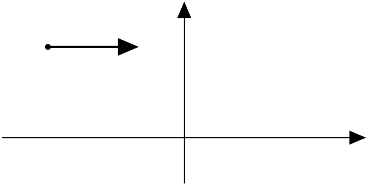

Figure: The x-axis is the beach, and the boat sails on the line $y = 4$.