Mathematical Analysis I: Lecture 23

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Start recording...

Annoucements

- Today: Apostol Vol. 1, Chapter 4.7,11-13.
- See Chapter 4.12 for concrete situations of differentiation.

Tangent line

We defined derivative as the limit of average slope of a graph, and expected that it should represent the slope at one point. If we have the slope at one point, then we should be able to draw the tangent line to the graph at that point.

Recall that the **slope** of a segment (x_0, y_0)–(x_1, y_1) is defined by $\frac{y_1 - y_0}{x_1 - x_0}$.

The graph of $y = Ax + B$ has the slope A. Therefore, if the graph of the function $y = f(x)$ passes the point (x_0, y_0) and the derivative is $f'(x_0)$, the tangent line should be

$$y = f'(x_0)(x - x_0) + y_0 = f'(x_0)x + y_0 - f'(x_0)x_0.$$

Indeed, this is of the form $y = Ax + B$ with $A = f'(x_0)$ and $B = y_0 - f'(x_0)x_0$, and passes the point (x_0, y_0).

Tangent line

When the slope is positive, the line goes upwards (when one goes to the right), while the line goes downwards when the slope is negative. When the slope is 0, it is a holizontal line. The vertical line is represented by the equation $x = a$, and this is not of the form $y = Ax + B$.

Tangent line

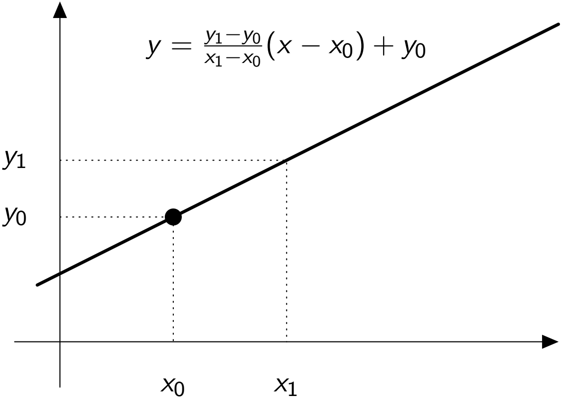

Figure: The slope of the straight line is $\frac{y_1 - y_0}{x_1 - x_0}$.

Tangent line

If we draw these lines, they are almost always indeed tangent, but in some cases they closs the graph.

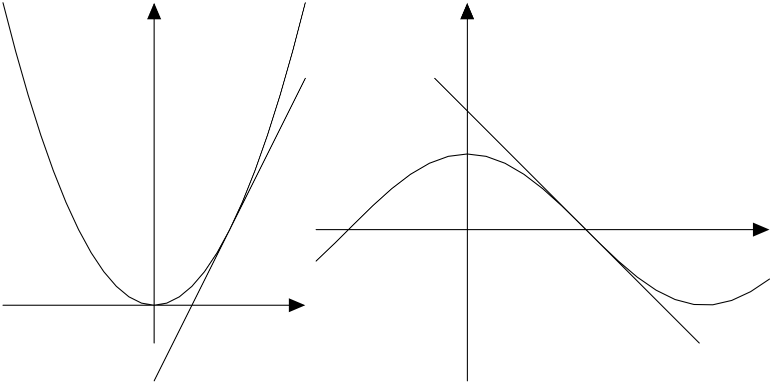

Figure: The tangent lines to the graphs of $x^2, \cos x$. Their equations are $y = 2(x-1)+1, y = -(x - \frac{\pi}{2})$, respectively.

Extrema and stationary points

Definition

Let $x \in \mathbb{R}$. For $\epsilon > 0$, we call the interval $(x - \epsilon, x + \epsilon)$ the ϵ**-neighbourhood of** x.
Let f be defined on an interval I. We say that f takes a **local minimum** or **relative minimum** (**local maximum** or **relative maximum** , respectively) at $x \in I$ if there is an $\epsilon > 0$ of x such that x is the minimum (maximum, respectively) of f in $(x - \epsilon, x + \epsilon) \cap I$.
If x is the minimum (maximum) of f on I we may say that x is the **global** or **absolute maximum** (**minimum**), to distinguish them from local (relative) minimum (maximum).

Example

Let $f(x) = x^3 - x$. When we consider this as a function on $\mathbb{R}$, there is no global maximum or mininum, but there are local maximum and minimum at $x = \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$, respectively (we will see why they are these points later). If we restrict the funcion to $[-2, 2]$, then $-2, 2$ are the global minimum and the global maximum, respectively.

Extrema and stationary points

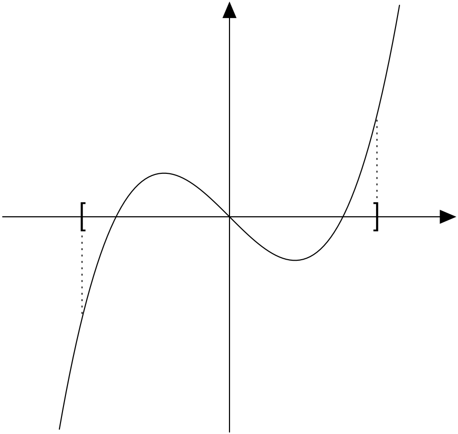

Figure: The graph of $y = x^3 - x$. The local maximum and minimum are $x = \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$, respectively. When restricted to a closed bounded interval, it has global maximum and minimum.

Extrema and stationary points

Theorem

Let f be defined on an open interval I and assume that f takes a local minimum (or a local maximum) at the point $c \in I$. If f is differentiable at c, then $f'(c) = 0$.

Proof.

Let c be a local maximum (the case for minimum is analogous). Then $f(x) \leq f(c)$ for all $x \in (c - \epsilon, c + \epsilon)$. As $f(x)$ is differentiable at $x = c$, both of its left and right derivatives must coincide.
On the other hand, $\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$, and $\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0$, therefore, $f'(c) = 0$. □

Extrema and stationary points

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A local minimum or a local maximum of a function f is called an **extremum**. A point x where $f'(x) = 0$ holds is called a **stationary point**.

Any extremum of a differentiable function is a stationary point by Theorem 3, but a stationary point is not necessarily an extremum.

Example

- $y = f(x) = x^3$.

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- $y = f(x) = x^3 - x$. Then $f'(x) = 3x^2 - 1$, hence $x = \pm\frac{1}{\sqrt{3}}$ are stationary points. They are local maximum and minimum, respectively.
- $y = |x|$. This function has the minimum at $x = 0$, but the function does not have derivative there. In particular, it does not hold $f'(0) = 0$ there ($f'(0)$ has no meaning there).

Extrema and stationary points

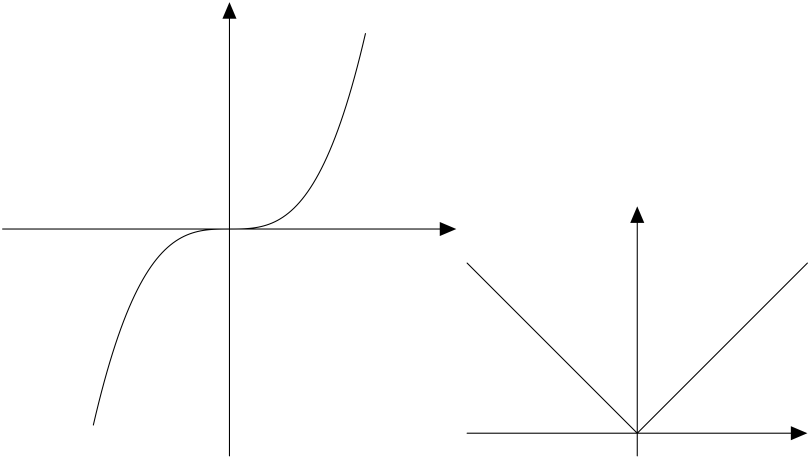

Figure: Left:The graph of $y = f(x) = x^3$. $f'(x) = 3x^2$, hence $x = 0$ is a stationary point. Right: The graph of $y = f(x) = |x|$. It has a minimum at $x = 0$ but it does not have a derivative

Implicit function

Imagine that we have a balloon and a gas is pumped into it at a rate of $50\mathrm{cm}^3/\mathrm{s}$. If the pressure remains constant, how fast is the radius of the balloon increasing when the radius is $5\mathrm{cm}$?

- The volume $V(t)$ of the balloon at time t (second): $V(t) = 50t\mathrm{cm}^3$. This implies $\frac{dV}{dt} = 50\mathrm{cm}^3/\mathrm{s}$.
- The radius $r(t)$ of the sphere with volume $V(t)$: $\frac{4\pi r(t)^3}{3} = V(t)$, By differentiating both sides by t, $4\pi\frac{dr}{dt}(t)r(t)^2 = \frac{dV}{dt}$.
- By solving this with $r(t_0) = 5$, $\frac{dr}{dt}(t_0) = \frac{50}{4\pi 5^2} = \frac{1}{2\pi}$.

Implicit function

Some shape can be represented by an equation, and the equation may define a function **implicitly**. For example, we know that the circle centered at $(0, 0)$ with radius r is given by

$$x^2 + y^2 = r^2.$$

As we saw before, if we consider only the part $y \geq 0$, it defines the function $y = \sqrt{r^2 - x^2}$.
It is not always possible to find an **explicit** expression for y of a given equation. Yet, an equation may define a function in an abstract way. Let us write it $y(x)$.

Implicit function

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It holds that $y(x)y'(x) = -x$.
This last relation can be also derived as follows: by taking the derivative of $x^2 + y(x)^2 = r^2$, we obtain $2x + 2y(x)y'(x) = 0$, hence $y(x)y'(x) = -x$.
If we know some concrete values of y, x (even if we do not know the general formula), then we can compute $y'(x)$ at that point.

The inverse trigonometric functions

$\sin x, \cos x, \tan x$ are injective on certain domains, and hence have the inverse functions. The standard choices are the following.

- $\sin x$: consider the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The range is $[-1, 1]$. The inverse function is denoted by $\arcsin x$, defined on $[-1, 1]$.
- $\cos x$: consider the interval $[0, \pi]$. The range is $[-1, 1]$. The inverse function is denoted by $\arccos x$, defined on $[-1, 1]$.
- $\tan x$: consider the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The range is $\mathbb{R}$. The inverse function is denoted by $\arctan x$, defined on $\mathbb{R}$.

Let us compute the derivative of $\arcsin y$ by putting $y = \sin x$. Then $D(\sin x) = \cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - y^2}$. By the general formula, $D(\arcsin y) = \frac{1}{D(\sin x)} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-y^2}}$.

Exercises

- Find the stationary points of $y = x^3 - 3x^2 + 3x$.
- Find the stationary points of $y = \sin(x^2)$.
- Consider the relation $y^2 - x^2 = 1$. Write y as an explicit function of x, and take the derivative. Differentiate it implicitly and find a relation.
- Consider the relation $y^5 + xy - 2x^3 = 0$. Check that $(x, y) = (1, 1)$ satisfy this equation. Assume that this defines an implicit function $y(x)$, and compute $y'(x)$ at $(1, 1)$.
- A boat sails parallel to a straight beach at a constant speed of 12 miles per hour, staying 4 miles offshore. How fast is it approaching a lighthouse on the shoreline at the instant it is exactly 5 miles from the lighthouse?