Mathematical Analysis I: Lecture 21

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Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 15:00-16:30 until 10th November. Then move to Tuesday morning.
- Today: Apostol Vol. 1, Chapter 4.1-4.

Derivative

As we discussed, we can define the average speed of a car, or the average slope of a curve in an interval. By taking the limit of the interval that tends to 0, we should obtain the speed or the slope at one point.

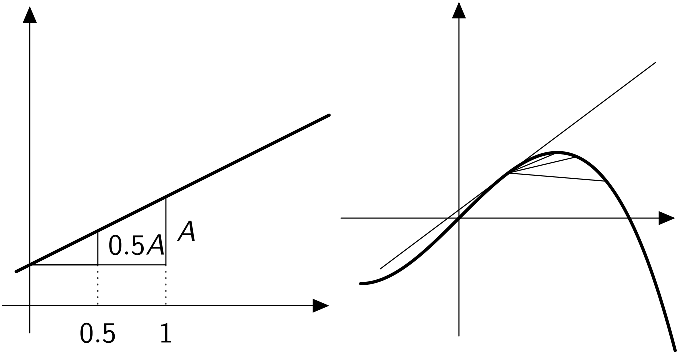

Figure: The slope of the straight lineat a point as the limit of the slopes of secant lines.

Definition

Let $I \subset \mathbb{R}$ an open interval, f a function defined on I.

- Let $x_0 \in I$ and h small such that $x_0 + h \in I$.

$$\frac{f(x_0+h) - f(x_0)}{h}$$

  is called the average rate of change of f between x_0 and $x_0 + h$.

- the function f is said to be **differentiable at** x_0 if the following limit exists:

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

  If this limit exists, it is called the **derivative of** f **at** x_0 and it is denoted by $f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$, $Df(x_0)$ or $\frac{df}{dx}(x_0)$.

The derivative at the point x_0 is defined to be the limit of average rates of change. In this sense, the derivative represents the rate of change af the point x_0. If $f(t)$ represents the position of a car at time t, then $f'(t)$ is the speed of the car at time t.

Derivative

- Let $f(x) = c$ for $x \in \mathbb{R}$ (constant). For any $x \in \mathbb{R}$, $\frac{f(x+h)-f(x)}{h} = \frac{c-c}{h} = 0$, therefore, $f'(x) = 0$.

- Let $A \in \mathbb{R}$ and $f(x) = Ax$ for $x \in \mathbb{R}$ (a straight line). For any $x \in \mathbb{R}$, $\frac{f(x+h)-f(x)}{h} = \frac{A(x+h)-Ax}{h} = \frac{Ah}{h} = A$, $f'(x) = A$.

- Let $A \in \mathbb{R}$ and $f(x) = Ax^2$ for $x \in \mathbb{R}$ (parabola). For any $x \in \mathbb{R}$, $\frac{f(x+h)-f(x)}{h} = \frac{A(x+h)^2-Ax^2}{h} = \frac{A(2xh+h^2)}{h} = A(2x+h)$, therefore, $f'(x) = \lim_{h\to 0} A(2x+h) = 2xA$.

Derivative

- Let $n \in \mathbb{N}$ and $f(x) = Ax^n$ for $x \in \mathbb{R}$. It holds that $(x+h)^n = \sum_{k=0}^n \binom{n}{k} x^k h^{n-k} = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \dots$. For any $x \in \mathbb{R}$,

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{A(x+h)^n - Ax^n}{h} \\ &= \frac{A(x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \dots h^n - x^n)}{h} \\ &= Anx^{n-1} + A \cdot \frac{n(n-1)}{2}x^{n-2}h + \cdots h^{n-1}, \end{aligned}$$

therefore,
$f'(x) = \lim_{h\to 0} A(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots h^{n-1}) = Anx^{n-1}$.

Derivative

Let $f(x) = \frac{1}{x}$ for $x \in \mathbb{R}, x \neq 0$. For any $x \in \mathbb{R}, x \neq 0$,

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{x - (x+h)}{hx(x+h)} = -\frac{1}{x(x+h)}$$

therefore, $f'(x) = \lim_{h \to 0} -\frac{1}{x(x+h)} = -\frac{1}{x^2}$.

Derivative

Let $f(x) = \log x$, $x > 0$. Then

$$\frac{\log(x+h) - \log x}{h} = \log\left(1 + \frac{h}{x}\right)^{\frac{1}{h}} = \frac{1}{x}\log\left(1 + \frac{h}{x}\right)^{\frac{x}{h}},$$

therefore, $f'(x) = \lim_{h\to 0} \frac{1}{x}\log(1+\frac{h}{x})^{\frac{x}{h}} = \lim_{y\to 0}\frac{1}{x}\log(1+y)^{\frac{1}{y}} = \frac{1}{x}$ (this is one of the notable limits we have learned)

Derivative

Let $f(x) = e^x$, $x \in \mathbb{R}$. Then

$$\frac{e^{x+h} - e^x}{h} = e^x \frac{e^h - 1}{h},$$

therefore, $f'(x) = \lim_{h \to 0} e^x \frac{e^h - 1}{h} = e^x$ (this is one of the notable limits).

Derivative

$f(x) = \sin x, x \in \mathbb{R}$. Recall the formula
$\cos\alpha \sin\beta = \frac{1}{2}(\sin(\alpha+\beta) - \sin(\alpha-\beta))$. Then, with $\alpha = x + \frac{h}{2}$, $\beta = \frac{h}{2}$, we have $\sin(x+h) - \sin x = 2\cos(x + \frac{h}{2})\sin\frac{h}{2}$, therefore,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{2\cos(x+\frac{h}{2})\sin\frac{h}{2}}{h} \\
&= \lim_{h\to 0} \cos\left(x + \frac{h}{2}\right) \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} \\
&= \cos x \cdot 1 = \cos x
\end{aligned}$$

(by the continuity of $\cos x$ and one of the notable limits $\lim_{h\to 0} \frac{\sin h}{h} = 1$ and the change of variable $\frac{h}{2}$ replacing h.

Derivative

$f(x) = \cos x, x \in \mathbb{R}$. Recall the formula $-\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha+\beta) - \cos(\alpha-\beta))$ Then, with $x + \frac{h}{2}$, $\beta = \frac{h}{2}$, we have $\cos(x+h) - \cos x = -2\sin(x+\frac{h}{2})\sin\frac{h}{2}$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\ &= \lim_{h\to 0}\frac{-2\sin(x+\frac{h}{2})\sin\frac{h}{2}}{h} \\ &= -\lim_{h\to 0}\sin\left(x+\frac{h}{2}\right)\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}} \\ &= -\sin x \cdot 1 = -\sin x \end{aligned}$$

(by the continuity of $\sin x$ and $\lim_{h\to 0}\frac{\sin h}{h} = 1$ and the change of variables).

Derivative

Lemma

If $f(x)$ is differentiable at x_0, then f is continuous at x_0.

Proof.

We compute the limit:

$$\begin{aligned}\lim_{x \to x_0} f(x) - f(x_0) &= \lim_{h \to 0} f(x_0 + h) - f(x_0) \\ &= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \cdot h \\ &= f'(x_0) \cdot 0 = 0.\end{aligned}$$

That is, $\lim_{x \to x_0} f(x) = f(x_0)$.

□

Derivative

Definition

Let $f : [x_0 - \delta, x_0] \to \mathbb{R}$ where $\delta > 0$. If the following limit $\lim_{h \to 0-} \frac{f(x_0+h)-f(x_0)}{h}$ exists (from the left), f is said to be left-differentiable at x_0, and this limit is denoted by $D_- f(x_0)$, the left derivative. Similarly, we define the right derivative.

Example

Let $f(x) = |x|, x_0 = 0$. $D_- f(0) = \lim_{h \to 0-} \frac{|0+h|-0}{h} = \lim_{h \to 0-} \frac{-h}{h} = -1$, while $D_+ f(0) = \lim_{h \to 0+} \frac{h}{h} = 1$.

Derivative

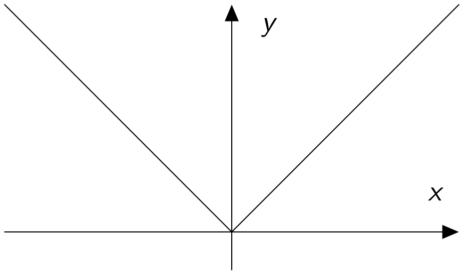

Figure: The graph of $y = |x|$, which has left and right derivatives, but they do not coincide.

Definition

Let f be defined on an open interval I. If f is differentiable at each point x of I, then $x \mapsto f'(x)$ defines a new function I. This is called the **derivative of** $f(x)$.

Example

- The derivative of $f(x) = C$ (constant) is $f'(x) = 0$.
- The derivative of $f(x) = x$ is $f'(x) = 1$.
- The derivative of $f(x) = x^2$ is $f'(x) = 2x$.
- The derivative of $f(x) = \sin x$ is $f'(x) = \cos x$.

Exercises

- Compute the derivative of $f(x) = \begin{cases} x^2 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$.
- Tell whether $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$ is continuous and is differentiable at $x = 0$.
- Tell whether $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$ is continuous and is differentiable at $x = 0$.
- Compute the derivative of $f(x) = x^3$.
- Compute the derivative of $f(x) = x^2 + x$.