Mathematical Analysis I: Lecture 20

Lecturer: Yoh Tanimoto

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Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 15:00-16:30 until 10th November. Then move to Tuesday morning.

Exercises

Compute $\cos\frac{5\pi}{4}, \sin\frac{7\pi}{3}, \sin\frac{115\pi}{4}, \sin(-\frac{23\pi}{3})$.

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Solution.

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Solution.

- $\cos\frac{5\pi}{4} = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$.

Compute $\cos\frac{5\pi}{4}, \sin\frac{7\pi}{3}, \sin\frac{115\pi}{4}, \sin(-\frac{23\pi}{3})$.
Solution.

- $\cos\frac{5\pi}{4} = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$.
- $\sin\frac{7\pi}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Compute $\cos\frac{5\pi}{4}, \sin\frac{7\pi}{3}, \sin\frac{115\pi}{4}, \sin(-\frac{23\pi}{3})$.
Solution.

- $\cos\frac{5\pi}{4} = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$.
- $\sin\frac{7\pi}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.
- $\sin\frac{115\pi}{4} = \sin\frac{3\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

Compute $\cos\frac{5\pi}{4}, \sin\frac{7\pi}{3}, \sin\frac{115\pi}{4}, \sin(-\frac{23\pi}{3})$.

Solution.

- $\cos\frac{5\pi}{4} = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$.
- $\sin\frac{7\pi}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.
- $\sin\frac{115\pi}{4} = \sin\frac{3\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.
- $\sin(-\frac{23\pi}{3}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$.

Compute $\cos\frac{\pi}{12}, \sin\frac{\pi}{12}, \sin\frac{\pi}{8}$.

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Solution.
Use $\cos^2 \theta = \frac{\cos 2\theta + 1}{2}, \sin^2 \theta = \frac{1 - \cos 2\theta}{2}$.

Compute $\cos\frac{\pi}{12}, \sin\frac{\pi}{12}, \sin\frac{\pi}{8}$.

Solution.

Use $\cos^2\theta = \frac{\cos 2\theta + 1}{2}, \sin^2\theta = \frac{1-\cos 2\theta}{2}$.

- $\cos\frac{\pi}{12} = \sqrt{\frac{\frac{\sqrt{3}}{2}+1}{2}}$.

Compute $\cos\frac{\pi}{12}, \sin\frac{\pi}{12}, \sin\frac{\pi}{8}$.

Solution.

Use $\cos^2\theta = \frac{\cos 2\theta + 1}{2}, \sin^2\theta = \frac{1-\cos 2\theta}{2}$.

- $\cos\frac{\pi}{12} = \sqrt{\frac{\frac{\sqrt{3}}{2}+1}{2}}$.
- $\sin\frac{\pi}{12} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$.

Compute $\cos\frac{\pi}{12}, \sin\frac{\pi}{12}, \sin\frac{\pi}{8}$.

Solution.

Use $\cos^2\theta = \frac{\cos 2\theta + 1}{2}, \sin^2\theta = \frac{1-\cos 2\theta}{2}$.

- $\cos\frac{\pi}{12} = \sqrt{\frac{\frac{\sqrt{3}}{2}+1}{2}}$.
- $\sin\frac{\pi}{12} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$.
- $\sin\frac{\pi}{8} = \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}}$.

Compute $\cos\frac{\pi}{4}, \sin\frac{\pi}{4}$ using $\cos\frac{\pi}{2} = 0$ and some of the general formulas.

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Solution.
Use $\cos^2\theta = \frac{\cos 2\theta + 1}{2}, \sin^2\theta = \frac{1-\cos 2\theta}{2}$.
$\cos\frac{\pi}{4} = \sqrt{\frac{\cos\frac{\pi}{2}+1}{2}} = \frac{1}{\sqrt{2}}, \sin\frac{\pi}{4} = \sqrt{\frac{1-\cos\frac{\pi}{2}}{2}} = \frac{1}{\sqrt{2}}$.

What is the domain of $\tan\theta$?

What is the domain of $\tan\theta$?

Solution.

$\tan\theta = \frac{\sin\theta}{\cos\theta}$, hence it is defined where $\cos\theta \neq 0$. $\cos\theta = 0$ if and only if $\theta = \frac{(2n+1)\pi}{2}$, hence $\tan\theta$ is defined for $\theta \neq \frac{(2n+1)\pi}{2}$.

Using the figure, explain the formula $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

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Solution.

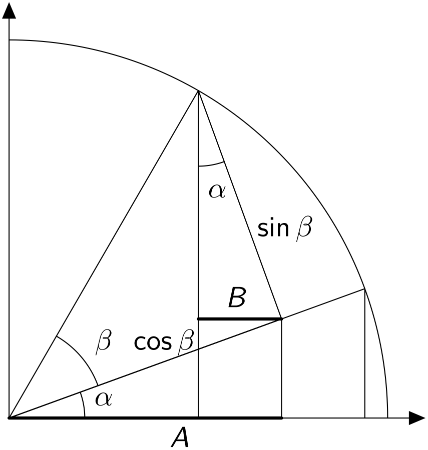

Figure: The formula $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.
$A = \cos\beta\cos\alpha$, $B = \sin\beta\sin\alpha$ and $A - B = \cos(\alpha+\beta)$.

Write $\cos 3\theta, \sin 3\theta$ in terms of $\cos\theta, \sin\theta$.

Write $\cos 3\theta, \sin 3\theta$ in terms of $\cos\theta, \sin\theta$.

Solution.

- $\cos 3\theta = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta = (\cos^2\theta - \sin^2\theta)\cos\theta - 2\cos\theta\sin^2\theta$.
- $\sin 3\theta = \sin 2\theta \cos\theta + \cos 2\theta \sin\theta = 2\cos^2\theta\sin\theta + (\cos^2\theta - \sin^\theta)\sin\theta$.

Prove that the union of open sets is open.

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Solution.

If $p \in \bigcup_{j\in J} A_j$ and A_j are open, then $p \in A_k$ for some $k \in J$ and there is $\epsilon > 0$ such that $(p - \epsilon, p + \epsilon) \subset A_k \subset \bigcup_{j\in J} A_j$, $\bigcup_{j\in J} A_j$ is open.

Prove that the intersection of closed sets is closed.

Prove that the intersection of closed sets is closed.

Solution.

If $a_n \in \bigcap_{j\in J} A_j$ and A_j are closed, then $a_n \in A_j$ for all $j \in J$ If $a_n \to a$, then $a \in A_j$ for all j because A_j is closed, hence $a \in \bigcap_{j\in J} A_j$ hence $\bigcap_{j\in J} A_j$ is closed.

Prove that the intersection of two open sets is open.

Prove that the intersection of two open sets is open.
Solution.
If $p \in A_1 \cap A_2$ and A_1, A_2 are open, then $p \in A_1, A_2$ and there are $\epsilon_1, \epsilon_2 > 0$ such that $(p - \epsilon_1, p + \epsilon_1) \subset A_1, (p - \epsilon_2, p + \epsilon_2) \subset A_2$. Let ϵ be the smallest of the two. Then $(p - \epsilon, p + \epsilon) \subset A_1 \cap A_2$, hence $A_1 \cap A_2$ is open.

Find an example of intersection of infinitely many open sets which is not open.

Find an example of intersection of infinitely many open sets which is not open.

Solution.

For example, consider $(-\frac{1}{n}, \frac{1}{n})$. It holds that $\bigcap_{n\in\mathbb{N}}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$. This is not open.

Find a subset of $\mathbb{R}$ which is both open and closed.

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Solution.
Let A be open and closed (and nonempty). Let $a \in A$. Consider $A^c \cap [a, \infty)$. This is bounded below, hence if it is not empty, there is $\inf(A^c \cap [a, \infty))$. If $x = \inf(A^c \cap [a, \infty)) \notin A^c$, then there is $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset A^c$ because A is closed (hence A^c is closed), hence there are points below x and in $A^c \cap [a, \infty)$, which contradicts that $x = \inf(A^c \cap [a, \infty))$. Hence $x \in A$. But then $(x - \epsilon, x + \epsilon) \subset A$ because A is open, which contradicts that $x = \inf(A^c \cap [a, \infty))$. Therefore, $A^c \cap [a, \infty)$ must be empty. Similarly, $A^c \cap (-\infty, a]$ is empty. That is, $A = \mathbb{R}$. Then indeed A is both open and closed.

Find a function, continuous defined on $\mathbb{R}$ but bounded.

Find a function, continuous defined on $\mathbb{R}$ but bounded.

Solution.

$\sin\theta, \cos\theta, \tanh x$, and so on.

Find a function, not continuous defined on $\mathbb{R}$ but bounded.

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Solution.

$\operatorname{sign} x, x - [x]$, and so on.

Tell whether $y = \cos x$ admits maxima and minima, and if so, list them up.

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Solution.
As $\cos^2 x + \sin^2 x = 1$, it holds that $-1 \leq \cos x \leq 1$. $\cos x = 1$ if and only if $x = 2n\pi, n \in \mathbb{Z}$. $\cos x = -1$ if and only if $x = (2n+1)\pi, n \in \mathbb{Z}$.

Tell whether $y = \tanh x$ admits maxima and minima, and if so, list them up.

Tell whether $y = \tanh x$ admits maxima and minima, and if so, list them up.
Solution.
As $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, this is monotonically increasing. Indeed,

$$\tanh x = \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

and if $x > y$, then $1 - e^{-2x} > 1 - e^{-2y}$ while $1 + e^{-2x} < 1 - e^{-2y}$, hence $\tanh x > \tan y$. This means that there is no maxima nor minima.

Tell whether $y = x$ is uniformly continuous or not, and prove it.

Tell whether $y = x$ is uniformly continuous or not, and prove it.
Solution.
For any $x \in \mathbb{R}$ and $\epsilon > 0$, we can take $\delta = \epsilon$, then for y such that $|y - x| < \delta = \epsilon$ we have $|f(y) - f(x)| = |y - x| < \epsilon$. Therefore, this is uniformly continuous.

Tell whether $y = x^2$ is uniformly continuous or not, and prove it.

Tell whether $y = x^2$ is uniformly continuous or not, and prove it.
Solution.
Let $\epsilon = 1$. For any $\delta > 0$, we can take $x > \frac{1}{\delta}$ then
$f(x+\delta) - f(x) = (x+\delta)^2 - x^2 = 2x\delta + \delta^2 > 2 > \epsilon$. Therefore, this is not uniformly continuous.

Tell whether $y = \sin x$ is uniformly continuous or not, and prove it.

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Solution.
By the Heine-Cantor theorem, $y = \sin x$ restricted to $[0, 4\pi]$ is uniformly continuous. That is, for any $\epsilon > 0$ there is $\delta > 0$ such that $|\sin(x) - \sin(y)| < \epsilon$ if $|x - y| < \delta, x, y \in [0, 4\pi]$. Then, for any $x, y \in \mathbb{R}$ such that $|x - y| < \delta$, there is n such that $x + 2n\pi, y + 2n\pi \in [0, 4\pi]$. Therefore, $|f(x) - f(y)| = |f(x + 2n\pi) - f(y + 2n\pi)| < \epsilon$. Therefore, this is uniformly continuous.

Tell whether $y = \tanh x$ is uniformly continuous or not, and prove it.

Tell whether $y = \tanh x$ is uniformly continuous or not, and prove it.
Solution.
Let $\epsilon > 0$.
We know that $\lim_{x\to\infty} \tanh x = 1, \lim_{x\to-\infty} \tanh x = -1$. Therefore, there is $M > 0$ such that $1 - \frac{\epsilon}{2} < \tan x < 1$ for $x > M$. Similarly, $-1 < \tan x < -1 + \frac{\epsilon}{2}$ for $x < -M$. On the other hand, on $[-M, M]$, $\tanh x$ is uniformly continuous, hence there is $\delta > 0$ such that if $|x - y| < \delta$ then $|\tanh x - \tanh y| < \frac{\epsilon}{2}$.
Then, for any two points x, y such that $|x - y| < \delta$, $|\tanh x - \tanh y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ by possibly taking the point in the middle M or $-M$. Therefore, this is uniformly continuous.