Mathematical Analysis I: Lecture 19

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Annoucements

- Today: Apostol Vol. 1, Chapter 3.16-17, 4.1.

The maximum and minimum of functions

Definition

Let f be a function defined on S.

- We say that f takes its **maximum** at x_0 if $f(x_0) \geq f(x)$ for all $x \in S$.
- We say that f takes its **minimum** at x_0 if $f(x_0) \leq f(x)$ for all $x \in S$

Example

Note that a function does not necessarily admit maximum or minimum. If it has, they may depend on the domain.

- $f(x) = x$, defined on $x > 0$, has no maximum or minimum. Indeed, for any $x > 0$, $f(\frac{x}{2}) = \frac{x}{2} < x$ and $f(2x) = 2x > x$.
- $f(x) = x^2$, defined on $x \in \mathbb{R}$, has no maximum but the minimum is at $x = 0$ with $f(0) = 0$. If it is restricted to the interval $[a, b]$, then the maximum is the larger one of a^2, b^2.

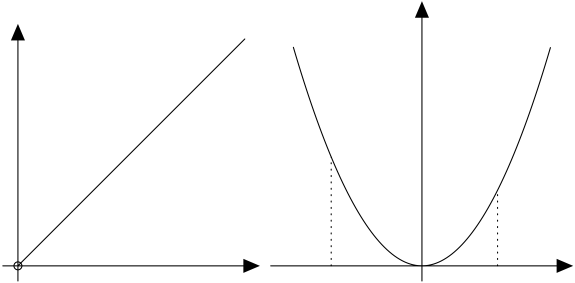

Figure: Left: $y = x$ on $x > 0$. There are no minimum or maximum. Right $y = x^2$ on $\mathbb{R}$. The minimum is 0 at $x = 0$, but there is not maximum. When restricted to $[a, b]$, either a^2 or b^2 is the maximum.

The maximum and minimum of functions

Theorem (Weierstrass)

Let $F \subset \mathbb{R}$ be a bounded closed set (or interval), and f be a continuous function on F. Then f admits both a maximum and a minimum in F.

Proof.

By a Theorem in the previous lecture, f is bounded, say $-M < f(x) < M$. Then the image $A = \{f(x) : x \in F\}$ is a bounded set in $\mathbb{R}$, therefore, it admits $\sup A$ and $\inf A$. Let us prove that f admits a maximum (the case for minimum is analogous). For each n there is $x_n \in F$ such that $\sup A - \frac{1}{n} < f(x_n)$.

As F is bounded, x_n admits a convergent subsequence y_n, $y_n \to y$ and $y \in F$ because F is closed. Now, as f is continuous, we have $f(y) = \lim_{n\to\infty} f(y_n)$. As y_n is a subsequence, it holds that $\sup A - \frac{1}{n} < f(y_n) \leq \sup A$. This implies that $f(y) = \sup A$. That is, f attains a maximum at y. □

The maximum and minimum of functions

Example

(and non example)

- $f(x) = x^2$ is continuous, hence on any closed and bounded F f admits a maximum and a minimum. But not on the whole real line $\mathbb{R}$, which is not bounded.
- $f(x) = x - [x]$ is not continuous, and indeed it does not admit a maximum on $[0, 1]$, although $[0, 1]$ is close and bounded.

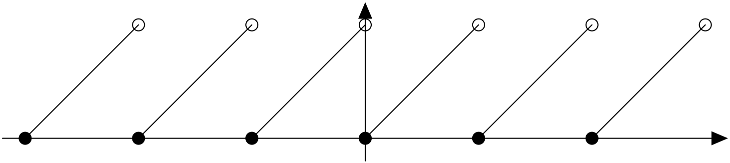

Figure: The graph of the function $y = x - [x]$, the decimal part of x. This is bounded, but has no maximum. The minimum is 0 at $x \in \mathbb{Z}$.

Compactness

Often it is said that a closed and bounded set $F \subset \mathbb{R}$ is **compact**. We have seen that any sequence $\{a_n\}$ in a compact set admits a convergent subsequence (the Bolzano-Weierstrass theorem), and the limit is in F. Conversely, if a set A has a property that any sequence in it has a convergent subsequence with the limit in A, then it is compact (bounded and closed): indeed, A must be bounded because otherwise we could take an unbounded sequence. Furthermore, A must be closed, because if $a_n \in A$ is a convergent sequence, we can take a convergent subsequence with the limit a in A, but there is only one limit for a_n, hence $a_n \to a \in A$, that is, A is closed.

Uniform continuity

Let us see another strong property of continuous functions defined on bounded and closed sets.

Definition

Let $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$. f is said to be **uniformely continuous on** S if, for any $\epsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in S, |x - y| < \delta$.

Note the difference with the continuity: a function f is continuous if for each $x \in S$ and for each ϵ there is δ such that $|f(y) - f(x)| < \epsilon$ if $|y - x| < \delta$. In other words, the number δ may change from point x to others.
On the other hand, uniform continuity asserts that for each $\epsilon > 0$ there is δ that applies to all $x, y \in S$, hence *uniformly* in S.

Uniform continuity

Example

(functions that are not uniformly continuous)

- $f(x) = \frac{1}{x}$ is continuous on $\{x \in \mathbb{R} : x > 0\}$. However, it is not uniformly continuous. Indeed, for $\epsilon = 1$ for any $\delta > 0$, we can take N such that $\frac{1}{N} < \delta$ and $N > 2$. Then $x = \frac{1}{N}, y = \frac{2}{N}$, hence $f(y) - f(x) = \frac{N}{2} > 1 = \epsilon$ but $x - y = \frac{1}{N} < \delta$.
- $f(x) = \sin\frac{1}{x}$ is continuous on $\{x \in \mathbb{R} : x > 0\}$. Indeed, for $\epsilon = \frac{1}{2}$ for any $\delta > 0$, we can take N such that $\frac{2}{\pi N} < \delta$ and N odd. Then $x = \frac{2}{\pi N}, y = \frac{1}{\pi N}$, hence $|f(\frac{1}{\pi N}) - f(\frac{2}{\pi N})| = |\sin(\pi N) - \sin(\frac{\pi N}{2})| = 1 > \epsilon$ but $x - y = \frac{1}{\pi N} < \delta$.

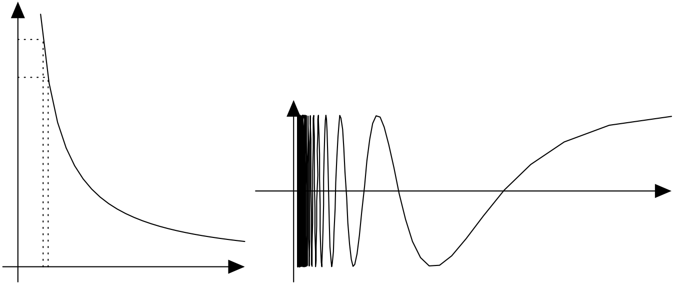

Figure: Functions continuous but not uniformly continuous.

Uniform continuity

Note that the function $f(x) = |x|$ is continuous. Indeed, if $x > 0$, then $f(x) = x$ and this is continuous at x. Similarly, f is continuous at $x < 0$. Finally, if $x = 0$, for any $\epsilon > 0$, we take $\delta = \epsilon$. Then if $|y - x| = |y - 0| < \delta$, then $|y| - |0| = |y - 0| < \delta = \epsilon$.

Theorem (Heine-Cantor)

Let F bounded and closed, $f : F \to \mathbb{R}$ a continuous function. Then f is uniformly continuous.

Uniform continuity

Proof.

To prove this by contradiction, assume that there is $\epsilon > 0$ such that for any $\delta > 0$ there are $x, y \in F, |x - y| < \delta$ but $|f(x) - f(y)| > \epsilon$. In particular, for $\delta = \frac{1}{n} > 0$ there are $x_n, y_n \in F$ such that $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| > \epsilon$. Let x_{N_n} be a convergent subsequence of x_n (which exists by the Bolzano-Weierstrass Theorem) to $\tilde{x} \in F$. Let us extract a subsequence $\{y_{N_n}\}$ of $\{y_n\}$. As $|\tilde{x} - y_{N_n}| \leq |\tilde{x} - x_{N_n}| + |x_{N_n} - y_{N_n}| \to 0$, also $\{y_{N_n}\}$ must be convergent to $\tilde{x} \in F$.
Then $\lim_{n\to\infty} |f(x_{N_n}) - f(y_{N_n})| = |f(\tilde{x}) - f(\tilde{x})| = 0$, as f is continuous (note that the absolute value is continuous). But this contradicts the assumption that $|f(x_{N_n}) - f(y_{N_n})| > \epsilon$.
Therefore, for all ϵ there exists δ such that for all $x, y \in F, |x - y| < \delta$ vale $|f(x) - f(y)| < \epsilon$. □

Uniform continuity

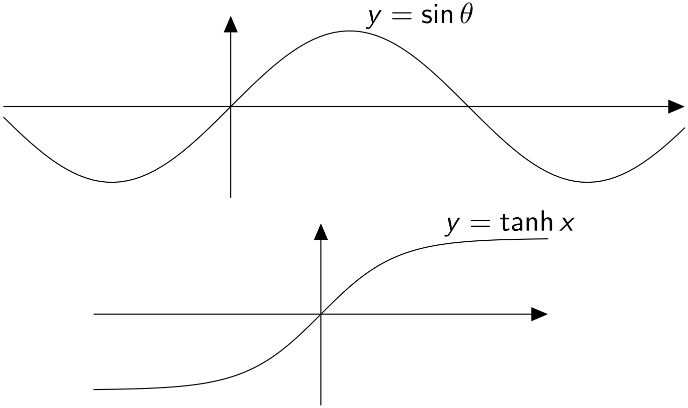

Figure: Functions defined on $\mathbb{R}$ but uniformly continuous.

Preliminaries for derivatives

Until now, we have studied continuity of functions. A function f is continuous at point x if for each $\epsilon > 0$ there is δ such that $|f(y) - f(x)| < \epsilon$ for y such that $|y - x| < \delta$. This tells us that "the graph is connected", but does not tell us how fast the function f changes.
We would like to know such information. For example, if f represents the motion of a car (in one direction), then how can we determine the **speed** of the car? Or if f represents the height of the mountain in a path and x represents the distance from the starting point, what is the **slope** of the mountain?
In the case of the speed, if the car has travelled $100\mathrm{km}$ in two hours, then the *average speed* is $50\mathrm{km}/\mathrm{h}$ per hour. But it might be that the car travelled with the constant speed $50\mathrm{km}/\mathrm{h}$, or it travelled with $40\mathrm{km}/\mathrm{h}$ in the first one hour and then $60\mathrm{km}/\mathrm{h}$ in the second one hour. Is it possible to determine the speed at a moment? In the case of a mountain, what is the slope at a point?
They should be approximated by secant lines.

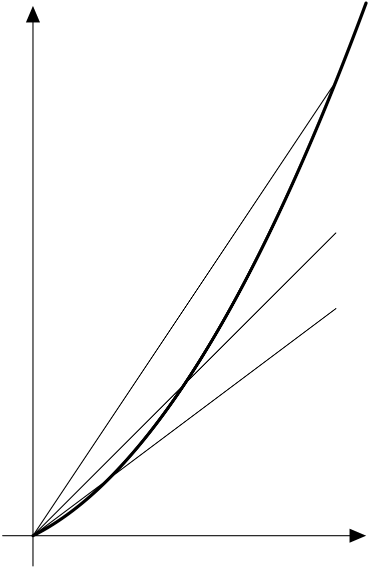

Figure: The slope at a point as the limit of the slopes of secant lines.

Exercises

- Tell whether $y = \cos x$ admits maxima and minima, and if so, list them up.
- Tell whether $y = \tanh x$ admits maxima and minima, and if so, list them up.
- Tell whether $y = x$ is uniformly continuous or not, and prove it.
- Tell whether $y = x^2$ is uniformly continuous or not, and prove it.
- Tell whether $y = \sin x$ is uniformly continuous or not, and prove it.
- Tell whether $y = \tanh x$ is uniformly continuous or not, and prove it.