Mathematical Analysis I: Lecture 18

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Start recording...

Annoucements

- There will be tutoring (exercise/question session by a Ph. D. student). Probably on Tuesday, but we start in the afternoon until the end of Basic Mathematics course, then move to the morning.
- If you are not sure with exponential functions and logarithm, or $\cos, \sin$, attend the **complementary class and do exercises**.
- Today: Apostol Vol. 1, Chapter 3.16

Open and closed sets

Definition

Let $O \subset \mathbb{R}$. We say that O is **open** if for any $p \in O$ there is $\epsilon > 0$ such that $(p - \epsilon, p + \epsilon) \subset O$ (this ϵ depends on p). Let $F \subset \mathbb{R}$. We say that F is **closed** if for any convergent sequence $\{a_n\} \subset F, a_n \to a$, it holds that $a \in F$.

Open and closed sets

Example

- Consider the open interval $A = (0, 1)$. This is open, because for any point $p \leq \frac{1}{2}$ we can take $\epsilon = \frac{p}{2}$ and $(\frac{p}{2}, \frac{3p}{2}) \subset (0, 1)$. If $p > \frac{1}{2}$, we can take $\epsilon = \frac{1-p}{2}$. On the other hand, $(0, 1)$ is not closed. Indeed, the sequence $a_n = \frac{1}{n}$ belongs to $A = (0, 1)$, but the limit 0 does not belong to A.
- Consider the closed interval $B = [0, 1]$. This is closed. Indeed, for any convergent sequence $\{a_n\} \subset B$, $a_n \to a$, it holds that $0 \leq a_n \leq 1$ and hence $0 \leq a \leq 1$. On the other hand, for $p = 0$, for any ϵ, $(-\epsilon, \epsilon) \not\subset B$, therefore, B is not open.

Therefore, the terminology “open” and “closed” for intervals are consistent with those for general sets we have just introduced.

Open and closed sets

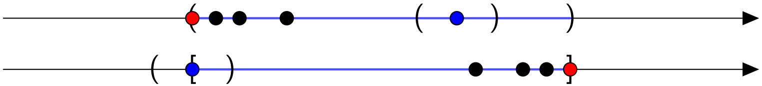

Figure: Open and closed intervals. An open set include a small "neighborhood" of any point in it, but a sequence in it might converge to a point outside. A closed subset contains the limit of any sequence in it, but a point might "touch" other points outside.

Open and closed sets

For any set $A \subset \mathbb{R}$, we denote its complement by $A^{\mathrm{c}} = \mathbb{R} \setminus A$.

Lemma

$O \subset \mathbb{R}$ is open if and only if O^{c} is closed.

Proof.

Let O be open and assume that O^{c} is not closed. That is, there is a sequence $\{a_n\} \subset O^{\mathrm{c}}$ that converges to a, but $a \in O^{\mathrm{c}}$. Therefore, it must holds $a \in O$. But we can take $\epsilon > 0$ such that $(a - \epsilon, a + \epsilon) \subset O$, and if $a_n \to a$, it would have to hold that $a_n \in (a - \epsilon, a + \epsilon) \subset O$, which contradicts the assumption that $\{a_n\} \subset O^{\mathrm{c}}$. Therefore, O^{c} is closed.
Conversely, let O^{c} be closed, and assume that O is not open. As O is not open, there is $a \in O$ such that for any $\frac{1}{n} > 0$ there is a_n such that $|a_n - a| < \frac{1}{n}$, but $a_n \notin O$. Hence $a_n \in O^{\mathrm{c}}$. But with this condition $a_n \to a$, which contradicts the assumption that O^{c} is closed. Therefore, O must be open. $\square$

Open and closed sets

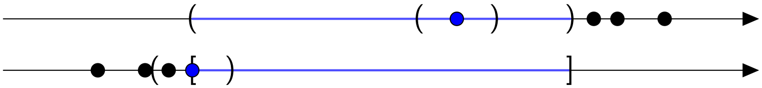

Figure: Any point in an open set is "protected" from outside. On the other hand, if a set is not open, there is a point which is not "protected".

Open and closed sets

It is not difficult to prove that any union (even if infinite!) of open sets is again open. Similarly, any intersection of closed sets is again closed.
Let us recall that a sequence $\{a_n\}$ is called Cauchy if for any given $\epsilon > 0$ there is N such that for $m, n > N$ it holds that $|a_m - a_n| < \epsilon$.
Furthermore, we said that b_n is a subsequene of a_n if there is a growing sequence $N_n \in \mathbb{N}$ such that $b_n = a_{N_n}$, that is, b_n is obtained by skipping some elements in a_n. Recall that we consider *infinite* sequences, that is, the sequence does not stop at any a_n, but continues infinitely.

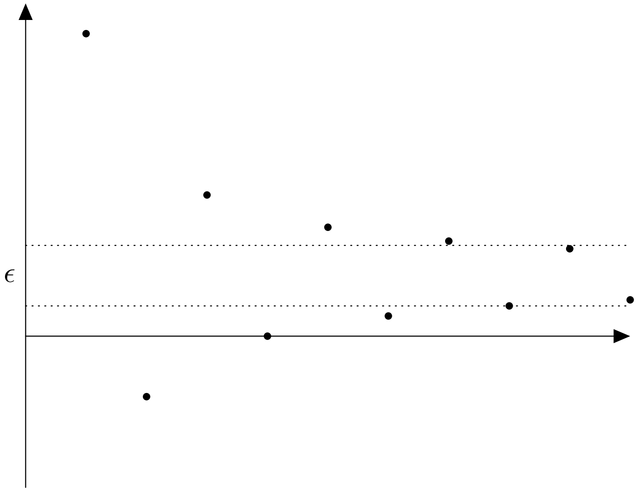

Figure: A Cauchy sequence.

Bolzano-Weierstrass theorem

Theorem (Bolzano-Weierstrass)

Let $\{a_n\}$ be a bounded sequence. Then there is a convergent subsequence of $\{a_n\}$.

Proof.

As $\{a_n\}$ is bounded, we can find M sufficiently large such that $a_n \in [-M, M]$. As the sequence $\{a_n\}$ infinitely many elements, one of the intervals $[-M, 0], (0, M]$ must contain infinitely many of them. Therefore, we can take a subsequence $b_n = a_{m_n}$ such that b_n are contained one of them. To fix the idea, assume that $b_n \in (0, M]$ (the other case is just analogous).

Bolzano-Weierstrass theorem

Proof.

As $(0, M] = (0, \frac{M}{2}] \cup (\frac{M}{2}, M]$, one of them must contain infinitely many elements of b_n. Therefore, we can take a subsequence $c_n = b_{k_n}$ such that c_n are contained one of them.

By continuing this procedure, for each n we obtain a subsequence that is contained in an interval of length $\frac{M}{2^{n-1}}$, and the later one is a subsequence of the former. Let us take a subsequence $a_1, b_2, c_3, \cdots$ of the original sequence. Then, for $n, m > N$, any two elements are contained in an interval of length $\frac{M}{2^{N-1}}$. Therefore, this subsequence is Cauchy. Then it is a convergent sequence by the property of Cauchy sequence. $\square$

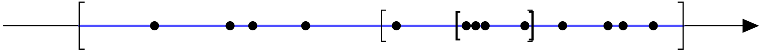

Figure: Nested invertals. As the sequence $\{a_n\}$ contains infinitely many points, one of two intervals must contain infinitely many of them.

Bolzano-Weierstrass theorem

It is important that a_n is bounded. Indeed, if not, it is obviously impossible in general to extract a convergent sequence: consider $a_n = n$, which is not bounded and not convergent to any point. In addition, the possibility to extract a convergent subsequence does not mean that the original sequence is convergent, or there is only one convergent subsequence.

Continuous functions on closed sets

Theorem

Let f be a continuous function defined on a bounded closed interval F. Then f is bounded, that is, there is $M > 0$ such that $|f(x)| < M$ for $x \in F$.

Proof.

Let us suppose the contrary, that for any $n > 0$ there is $x_n \in F$ such that $|f(x_n)| \geq n$. As $\{x_n\}$ is a sequence in a bounded set F, we can take a convergent subsequence $\{y_n\}$ of $\{x_n\}$. As F is closed, $y_n \to y$ and $y \in F$. By assumption f is continuous, therefore, it must hold that $\lim_{n\to\infty} f(y_n) = f(y)$. But this is impossible because $|f(y_n)| \geq n$ by our choice. Therefore, f is bounded. $\square$

Continuous functions on closed sets

Example

- Consider the function $f(x) = \frac{1}{x}$ defined on $\mathbb{R} \setminus \{0\}$. This is not bounded, but when we restrict it to an interval $[\frac{1}{n}, n]$, it is bounded by n.
- Consider the function $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in [-1, 1], x \neq 0 \\ 0 & x = 0 \end{cases}$. This is defined on a closed interval $[-1, 1]$, but not continuous. Therefore, the previous theroem does not apply. Indeed, it is not bounded.

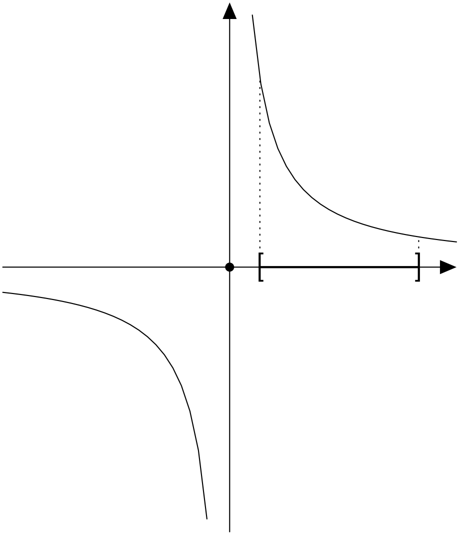

Figure: A continuous function on a bounded closed interval is bounded. If either of these conditions are violated, then function can be unbounded.

Exercises

- Find a subset of $\mathbb{R}$ which is both open and closed.
- Prove that the union of open sets is open.
- Prove that the intersection of closed sets is closed.
- Prove that the intersection of two open sets is open.
- Find an example of intersection of infinitely many open sets which is not open.
- Find a function, continuous defined on $\mathbb{R}$ but bounded.
- Find a function, not continuous defined on $\mathbb{R}$ but bounded.