Mathematical Analysis I: Lecture 17

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19/10/2020

Start recording...

Annoucements

- There will be tutoring (exercise/question session by a Ph. D. student). **Which day do you prefer?** Please answer in the form.
- If you are not sure with exponential functions and logarithm, or $\cos, \sin$, attend the **complementary class and do exercises**.
- Today: Apostol Vol. 1, Chapter 2.5, 2.7

Trigonometric functions

The functions $\sin\theta$ and $\cos\theta$ are usually defined as the length of the horizontal and vertical sides of the right triangle obtained from a point p on the unit circle (the circle centered at $(0,0)$ with radius 1) such that the x-axis and the segment from the point of origin to p makes an angle of degree θ. However, to make this definition precise, we would first need to define the **angle**, that is the **length of the arc** on the unit circle, then consider the right triangle...
That is possible, but we would have to wait until we define integral before define trigonometric functions (or define the trigonometric function by something called power series).

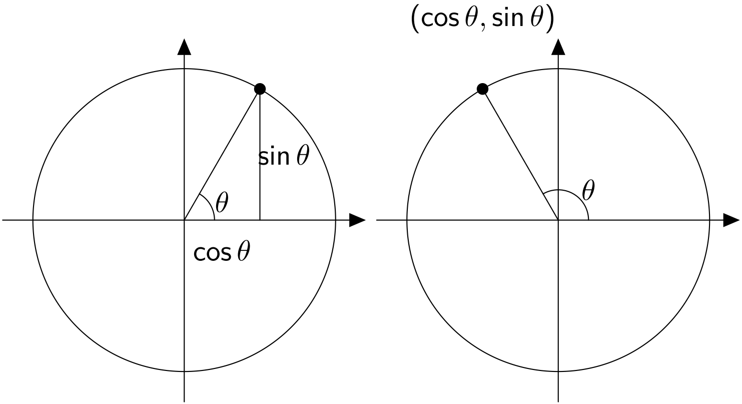


Figure: The trigonometric functions and their values for general angle θ.

Trigonometric functions

In this lecture, we prefer practicality, therefore,

- We assume that there are functions called $\sin\theta, \cos\theta$.
- We use the figures and the elementary geometry to derive their elementary properties.
- Then we study their analytic aspects: limit, derivative, integral, Taylor expansion, and so on.

Now, to obtain $\cos\theta$ and $\sin\theta$, we draw the unit circle, and take the point p on the unit circle such that the x-axis and the segment from the point of origin to p makes an angle of degree θ going **anticlockwise**, $0 \le \theta \le 90$ (degrees). Then $\cos\theta$ is defined to be the x-coordinate of the point p, and $\sin\theta$ is defined to be the y-coordinate of p.
We can make a right triangle by drawing the vertical line from this point. If $0 \le \theta \le 90$ (degrees), then $\cos\theta$ is the length of the horizontal side of the triangle, while $\sin\theta$ defined to be the length of the vertical side. When $\theta \ge 90$ (degrees), then $\cos\theta$ becomes negative.

Trigonometric functions

There are various ways to represent the angle. Often we use the **degrees**, which devide the circle into 360 degrees. Another is called the **radian**, which defines the angle by the lenght of the arc on the unit circle. In radian, we have 360 (degrees) $= 2\pi$ (radian), 180 (degrees) $= \pi$ (radian), 90 (degrees) $= \frac{\pi}{2}$ (radian), 45 (degrees) $= \frac{\pi}{4}$ (radian) and so on. In this lecture, from this point **we use radian**, unless otherwise specified.
Some important values:

- $\sin 0 = 0, \cos 0 = 1$.
- $\sin \frac{\pi}{6} = \frac{1}{2}, \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$.
- $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}, \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$.
- $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}, \cos \frac{\pi}{3} = \frac{1}{2}$.
- $\sin \frac{\pi}{2} = 1, \cos \frac{\pi}{2} = 0$.

Trigonometric functions

We can extend $\cos\theta$ and $\sin\theta$ to all real numbers, considering that for $\theta > 2\pi$ we go around the circle more than once, and for $\theta < 0$ we go around the circle clockwise. With this understanding, we have

- $\cos(\theta + 2\pi) = \cos\theta$
- $\sin(\theta + 2\pi) = \sin\theta$.
- $\cos(-\theta) = \cos\theta$
- $\sin(-\theta) = -\sin\theta$.

In this way, we can consider cos and sin as **functions** on $\mathbb{R}$. They are continuous, because if we change slightly the degree, the point p moves only slightly (we do not prove this, as we introduce these functions only by geometry, without defining the arg length).
They are related by the formulas $\cos(\theta + \frac{\pi}{2}) = -\sin\theta$ and $\sin(\theta + \frac{\pi}{2}) = \cos\theta$.
We introduce also $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

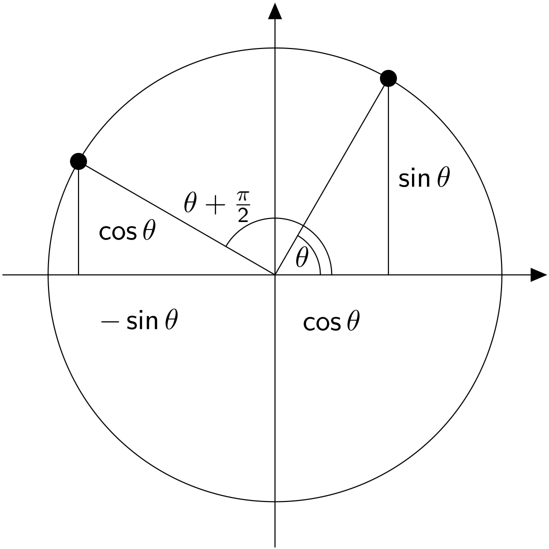
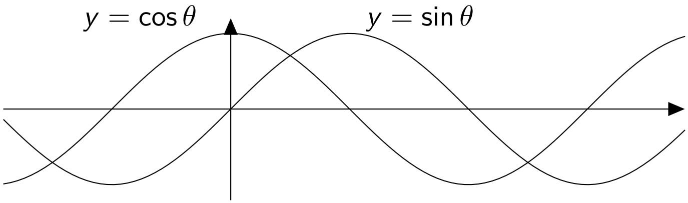
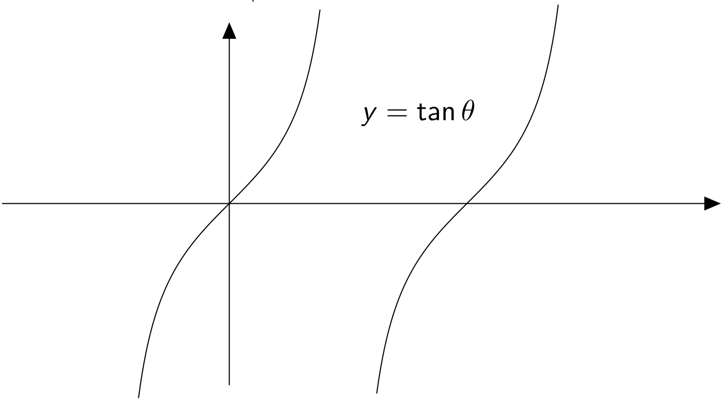


Figure: A relation $\cos(\theta + \frac{\pi}{2}) = -\sin\theta$ and $\sin(\theta + \frac{\pi}{2}) = \cos\theta$.


$y = \cos\theta$
$y = \sin\theta$


$y = \tan\theta$

Some formulas

We often write
$\cos^2\theta = (\cos\theta)^2, \sin^2\theta = (\sin\theta)^2, \cos^3\theta = (\cos\theta)^3, \sin^3\theta = (\sin\theta)^3$, etc.

- $\cos^2\theta + \sin^2\theta = 1$. This is because of the Pytagorean theorem: $\cos\theta$ and $\sin\theta$ are the length of the horisontal and vertical sides of the right triangle, while the length of the longest side is 1.
- $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.
- $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

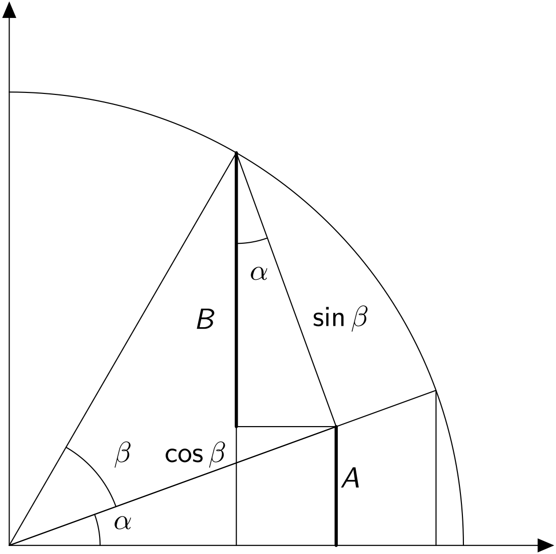


Figure: The formula $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.
$A = \cos\beta\sin\alpha$, $B = \sin\beta\cos\alpha$ and $A + B = \sin(\alpha+\beta)$.

Some formulas

From these formulas, we can derive various useful formulas.

- $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$. Indeed, $\cos 2\theta = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$ and use $\cos^2\theta + \sin^2\theta = 1$.
- $\sin 2\theta = 2\sin\theta\cos\theta$. Indeed, $\sin 2\theta = \sin\theta\cos\theta + \cos\theta\sin\theta$.
- $\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha+\beta) - \sin(\alpha-\beta))$. Indeed,

$$\begin{aligned}
&\frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta)) \\
&= \frac{1}{2}((\sin\alpha\cos\beta + \cos\alpha\sin\beta) + (\sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta))) \\
&= \frac{1}{2}(\sin\alpha\cos\beta + \cos\alpha\sin\beta + \sin\alpha\cos\beta - \cos\alpha\sin\beta) \\
&= \sin\alpha\cos\beta.
\end{aligned}$$

- $\cos\alpha\sin\beta = \frac{1}{2}(\sin(\alpha+\beta) - \sin(\alpha-\beta))$.
- $\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta))$.
- $\sin\alpha\sin\beta = \frac{1}{2}(-\cos(\alpha+\beta) + \cos(\alpha-\beta))$.

Some formulas

For example, we can compute $\cos \frac{\pi}{8}$.

Some formulas

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Indeed, $2\cos^2\frac{\pi}{8} - 1 = \cos(\frac{\pi}{8}\cdot 2) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, and hence
$\cos\frac{\pi}{8} = \sqrt{\frac{\frac{1}{\sqrt{2}}+1}{2}}$.

Some limit

By comparing the areas of the triangles of the sector, we see $\frac{1}{2}\cos\theta\sin\theta < \frac{\theta}{2} < \frac{1}{2}\frac{\sin\theta}{\cos\theta}$ (see Figure 4), and hence $\cos\theta < \frac{\sin\theta}{\theta} < \frac{1}{\cos\theta}$.
As we assumed that sin and cos are continuous, and $\cos 0 = 1$, we obtain $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \lim_{\theta\to 0}\cos\theta = \lim_{\theta\to 0}\frac{1}{\cos\theta} = 1$ by squeezing.

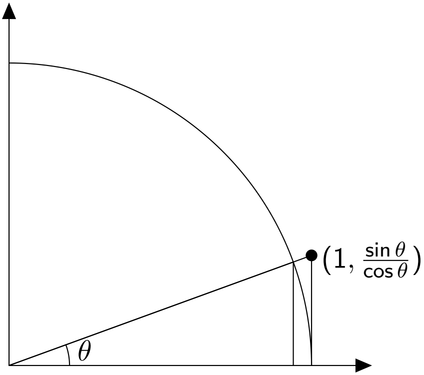

Figure: By comparing the areas of the triangles and the sector, we see $\frac{1}{2}\cos\theta\sin\theta < \frac{\theta}{2} < \frac{1}{2}\frac{\sin\theta}{\cos\theta}$.

Exercises

- Compute $\cos\frac{5\pi}{4}, \sin\frac{7\pi}{3}, \sin\frac{115\pi}{4}, \sin(-\frac{23\pi}{3})$.
- Compute $\cos\frac{\pi}{12}, \sin\frac{\pi}{12}, \sin\frac{\pi}{8}$.
- Compute $\cos\frac{\pi}{4}, \sin\frac{\pi}{4}$ using $\cos\frac{\pi}{2} = 0$ and some of the general formulas.
- What is the domain of $\tan\theta$?
- Using the figure, explain the formula $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.
- Write $\cos 3\theta, \sin 3\theta$ in terms of $\cos\theta, \sin\theta$.