Mathematical Analysis I: Lecture 16

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Exercises

Prove that for $p, q, r, s \in \mathbb{N}$, we have $(a^{\frac{p}{q}})^{\frac{r}{s}} = a^{\frac{pr}{qs}}$.

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Solution. We have

$$((a^{\frac{p}{q}})^{\frac{r}{s}})^{qs} = (a^{\frac{p}{q}})^{qr} = ((a^{\frac{p}{q}})^{q})^{r} = a^{pr},$$

and hence by taking the qs-th root of both sides we obtain the claim.

Let $a_n = \frac{(-1)^n}{n}$. Determine $\sup\{a_k : k \geq n\}$ and $\inf\{a_k : k \geq n\}$.

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Solution. Note that $a_n \geq 0$ if n is even, and $a_n < 0$ if n is odd. In addition, $|a_n| = \frac{1}{n}$ is monotonically decreasing.
If n is even, then a_n is the largest in $\{a_k : k \geq n\}$, hence $\sup\{a_k : k \geq n\} = \frac{1}{n}$, while the smallest element is a_{n+1}, hence $\inf\{a_k : k \geq n\} = -\frac{1}{n+1}$.
Similarly, if n is odd, then $\sup\{a_k : k \geq n\} = \frac{1}{n+1}$, while $\inf\{a_k : k \geq n\} = -\frac{1}{n}$.

Compute 2^x for $x = 1, 2, 3, 4, \frac{1}{2}, -\frac{3}{2}$.

Compute 2^x for $x = 1, 2, 3, 4, \frac{1}{2}, -\frac{3}{2}$.
Solution. $2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16, 2^{\frac{1}{2}} = \sqrt{2}, 2^{-\frac{3}{2}} = \frac{1}{2\sqrt{2}}$.

Compute $(\frac{1}{9})^x$ for $x = 1, 2, -3, -\frac{1}{2}, \frac{3}{2}$.

Compute $(\frac{1}{9})^x$ for $x = 1, 2, -3, -\frac{1}{2}, \frac{3}{2}$.

Solution. $(\frac{1}{9})^1 = \frac{1}{9}, (\frac{1}{9})^2 = \frac{1}{81}, (\frac{1}{9})^{-3} = 729, (\frac{1}{9})^{-\frac{1}{2}} = 3, (\frac{1}{9})^{\frac{3}{2}} = \frac{1}{27}$.

Imagine that there is a pond and the leaves of lotus doubles each day. If the pond is completely filled on day 100, when is the pond half filled?

Imagine that there is a pond and the leaves of lotus doubles each day. If the pond is completely filled on day 100, when is the pond half filled?
Solution. It's day 99, because on the next day the pond is filled completely.

Compute $\log_3(81), \log_{81} 3, \log_2 0.125$.

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Solution. $81 = 3^4$, hence $\log_3 81 = 4$. $3 = 81^{\frac{1}{4}}$, hence $\log_{81} 3 = \frac{1}{4}$. $0.125 = \frac{1}{8} = 2^{-3}$, hence $\log_2 0.125 = -3$.

Compute $(1 + \frac{1}{3})^3$.

Compute $(1 + \frac{1}{3})^3$.
Solution. $(\frac{4}{3})^3 = \frac{64}{27} = 2.370370...$
$(1 + \frac{1}{5})^5 = 2.48832$.
$(1 + \frac{1}{10000})^{10000} = 2,718145927$.
The true value $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n = 2.718281828....$

If $y = Ce^{ax}$, what is the relation between $z = \log y$ and x?

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Solution. We have $e^z = y$, hence $e^z = Ce^{ax}$, and by taking log, we have $z = ax + \log C$.

If $y = Cx^p$, what is the relation between $z = \log y$ and $w = \log x$?

If $y = Cx^p$, what is the relation between $z = \log y$ and $w = \log x$?
Solution. We have $e^z = y, e^w = x$, hence $e^z = Ce^{pw}$, and by taking log, we have $z = pw + \log C$.

Calculate the integer part of $\log_{10}(232720)$.

Calculate the integer part of $\log_{10}(232720)$.
Solution. Note that $10^5 = 100000 = 232720 < 1000000 = 10^6$. As $\log_{10} x$ is monotonically increasing, $5 < \log_{10} 232720 < 6$. Therefore, its integer part is 5.

Calculate the integer part of $\log_2(13567)$.

Calculate the integer part of $\log_2(13567)$.
Solution. Note that $2^{13} = 8192 < 13567 < 16384 = 2^{14}$. As $\log_2 x$ is monotonically increasing, $13 < \log_2 13567 < 14$. Therefore, its integer part is 13.

Compute $\lim_{n\to\infty}(1+\frac{1}{n})^{n^2}$.

Compute $\lim_{n\to\infty}(1+\frac{1}{n})^{n^2}$.
Solution. We know that $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$. In particular, for sufficiently large n, we have $(1+\frac{1}{n})^n > 2$, and hence $(1+\frac{1}{n})^{n^2} > 2^n \to \infty$.

Compute $\lim_{x\to 0} \frac{\log_a(1+x)}{x}$.

Compute $\lim_{x\to 0} \frac{\log_a(1+x)}{x}$.
Solution. Use the change of base $\log_a(1+x) = \log_a e \log(1+x)$, and hence

$$\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e \lim_{x\to 0} \frac{\log(1+x)}{x} = \log_a e.$$

Compute $\lim_{x \to 0} \frac{a^x - 1}{x}$.

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Solution. Use the change of vaiables: $a^x = e^{(\log a)x}$, and if $x \to 0$, then $(\log a)x \to 0$. Therefore,

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{x \to 0} \frac{e^{(\log a)x} - 1}{x}$$
$$= \lim_{x \to 0} \frac{e^{(\log a)x} - 1}{(\log a)x} \cdot \log a = \lim_{y \to 0} \frac{e^y - 1}{y} \cdot \log a = \log a.$$

Compute $\lim_{x \to 0} \frac{\sinh x}{x}$.

Compute $\lim_{x\to 0} \frac{\sinh x}{x}$.
Solution.

$$\lim_{x\to 0} \frac{\sinh x}{x} = \lim_{x\to 0} \frac{e^x - e^{-x}}{2x} = \lim_{x\to 0} \frac{e^x - 1 + 1 - e^{-x}}{2x}$$
$$= \lim_{x\to 0} \frac{e^x - 1}{2x} + \frac{e^{-x} - 1}{-2x} = \frac{1}{2} + \frac{1}{2} = 1.$$

Compute $\lim_{x\to\infty} \tanh x$.

Compute $\lim_{x\to\infty} \tanh x$.

Solution.

$$\lim_{x\to\infty} \frac{\sinh x}{\cosh x} = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1.$$

Compute $\lim_{x \to 0} \frac{\sinh x}{e^x - 1}$.

Compute $\lim_{x\to 0} \frac{\sinh x}{e^x - 1}$.
Solution.

$$\lim_{x\to 0} \frac{\sinh x}{e^x - 1} = \lim_{x\to 0} \frac{\sinh x}{x} \frac{x}{e^x - 1} = 1 \cdot 1 = 1.$$

Compute $\lim_{x \to 0} \frac{(1+x)^a - 1}{x}$.

Compute $\lim_{x\to 0} \frac{(1+x)^a-1}{x}$.
Solution.

$$\lim_{x\to 0} \frac{(1+x)^a - 1}{x} = \lim_{x\to 0} \frac{e^{a\log(1+x)} - 1}{x} = \lim_{x\to 0} \frac{e^{a\log(1+x)} - 1}{\log(1+x)} \frac{\log(1+x)}{x}$$

$$= \lim_{y\to 0} \frac{e^{ay} - 1}{y} \lim_{x\to 0} \frac{\log(1+x)}{x} = a.$$