Mathematical Analysis I: Lecture 15

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15/10/2020

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Some notable limits

Theorem

Let $a \in \mathbb{R}$. The function $f(x) = x^a$ defined on $\mathbb{R}_+$ satisfies $x^a y^a = (xy)^a$ and is continuous.

Proof.

Note that these properties hold if $f(x) = x^q$, where q is rational.
Let $x, y > 0$. For a rational q we have $(xy)^q = x^q y^q$ and hence by taking $q_n \to a$ we have $(xy)^a = x^a y^a$. As for continity, assume that $x \neq y$, then take $a < q \in \mathbb{Q}$. We have
$|f(y) - f(x)| = x^a |y^a x^{-a} - 1| = x^a \left|\left(\frac{y}{x}\right)^a - 1\right| < x^a \left|\left(\frac{y}{x}\right)^q - 1\right|$ and $\lim_{y \to x} \left|\left(\frac{y}{x}\right)^q - 1\right| = 1$ by the continuity of the rational case. Therefore, by squeezing we have $\lim_{y \to x} f(y) = f(x)$. □

Some notable limits

Let $L \in \mathbb{R}$, and f is a function defined on (a, ∞). If for each $\epsilon > 0$, there is X such that $|f(x) - L| < \epsilon$ for $x > X$, then we write that $\lim_{x \to \infty} f(x) = L$.

Example

$\lim_{x \to \infty} \frac{1}{x} = 0$. $\lim_{x \to \infty} \frac{x}{x-1} = 1$.

Some notable limits

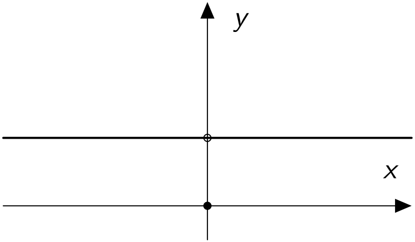
Let $f(x)$ defined on (a, b) and $L \in \mathbb{R}$. If for each $\epsilon > 0$ there is δ such that $|f(x) - L| < \epsilon$ for $x \in (a, a + \delta)$, we denote it by $\lim_{x\to a^+} f(x)$, and we call it the **right limit** of f at a. Similarly, we write $\lim_{x\to b^-} f(x)$ for the **left limit**.

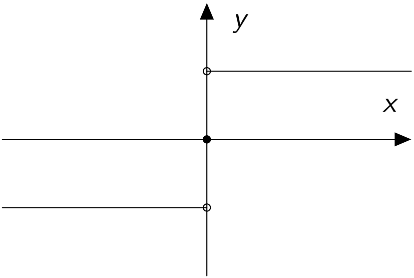
Example

Let $f(x) = \operatorname{sign} x$. $\lim_{x\to 0^+} f(x) = 1$, $\lim_{x\to 0^-} f(x) = -1$.

If $f(x)$ is defined on $(b, a) \cup (a, c)$, $\lim_{x\to a} f(x) = L$ exists if and only if both the left and right limits exist and $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$ and it is L. We leave the proof as an exercise.

Left and right limits





Some notable limits

Lemma

Let f be a function on S, $\lim_{y\to y_0} f(y) = L$. Assume that g is a function on T, continuous at x_0 and $g(x_0) = y_0$. Then $\lim_{x\to x_0} f(g(x)) = \lim_{y\to y_0} f(y) = L$. Similarly, if $\lim_{y\to\infty} f(y) = L$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} f(g(x)) = L$.

Proof.

The first statement can be proven similarly to the continuity of the composed function $f(g(x))$.
As for the second point, for a given ϵ we take Y such that $|f(y) - L| < \epsilon$ for $y > Y$. Then, there is X such that $g(x) > Y$ for $x > X$. Altogether, $|f(g(x)) - L| < \epsilon$ if $x > X$. □

We call this the change of variables, in the sense that we can calculate $\lim_{y\to y_0} f(y)$ by calculating $\lim_{x\to x_0} f(g(x))$ and vice versa.

Some notable limits

For $x \in \mathbb{R}$, we denote by $[x]$ the largest integer n such that $n \leq x$, and call it the **integer part of** x. For example, $[\sqrt{2}] = 1, [\pi] = 3$, and so on.
In the following, $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Theorem

We have the following.

1. $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = e^{-1}$.
2. $\lim_{n\to\infty} \left(1 + \frac{1}{n^2}\right)^n = 1$.
3. $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$.

Some notable limits

Proof.

- Note that

$$\left(1-\frac{1}{n}\right)^n = \left(\frac{n-1}{n}\right)^n = \left(1+\frac{1}{n-1}\right)^{-n}$$
$$= \left(1+\frac{1}{n-1}\right)^{-1}\frac{1}{\left(1+\frac{1}{n-1}\right)^{n-1}}$$

Note that $\frac{1}{x}$ is continuous at $x = 1, e$, and hence $\left(1+\frac{1}{n-1}\right)^{-1} \to 1$ and $\frac{1}{(1+\frac{1}{n-1})^{n-1}} \to \frac{1}{e}$. Altogether, $\left(1-\frac{1}{n}\right)^n = \frac{1}{e} = e^{-1}$.

□

Some notable limits

Proof.

- $\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^n = \lim_{n\to\infty}\left(\left(1+\frac{1}{n^2}\right)^{n^2}\right)^{\frac{1}{n}}$. As $\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{n^2} = e$, this sequence is bounded by, say M. Then $1 < \left(1+\frac{1}{n^2}\right)^n < M^{\frac{1}{n}}$ but $M^{\frac{1}{n}} \to 1$, then by squeezing we have $\left(1+\frac{1}{n^2}\right)^n \to 1$.

□

Some notable limits

Proof.

- Note that, if $a_n \to a$, then $b_n = a_{n+1} \to a$ as well. Furthermore, if $a < b < c$ and if $|a - x| < \epsilon, |c - x| < \epsilon$, then by the triangle inequality we have $-\epsilon < a - x < \epsilon$, hence $a - \epsilon < x < a + \epsilon$. Similarly, $c - \epsilon < x < c + \epsilon$, and therefore, $b - \epsilon < x < b + \epsilon$ and hence $|b - x| < \epsilon$.

  We know that $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1 + \frac{1}{n+1}\right)^{n+1} = e$. Let $n = [x]$, then $n \leq x < n + 1$ and

$$\frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{1 + \frac{1}{n+1}} < \left(1 + \frac{1}{x}\right)^x < \left(1 + \frac{1}{n}\right)^{n+1} = \left(1 + \frac{1}{n}\right)^n \cdot \left(1 + \frac{1}{n}\right).$$

  Note that the left-hand side and the right-hand side tend to e, because $1 + \frac{1}{n+1} \to 1, 1 + \frac{1}{n} \to 1$.

□

Proof.

This means that, for a given ϵ,
$\left|\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{1+\frac{1}{n+1}} - e\right| < \epsilon, \left|\left(1+\frac{1}{n}\right)^n \cdot \left(1+\frac{1}{n}\right) - e\right| < \epsilon$ for sufficiently large n. This implies that $\left|\left(1+\frac{1}{x}\right)^x - e\right| < \epsilon$.
Altogether, this says that, if x is sufficiently large, then we apply this argument with $n = [x]$, and obtain that $\left|\left(1+\frac{1}{x}\right)^x - e\right| < \epsilon$. This is $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$.

□

Some notable limits

Theorem

We have the following.

- $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$.
- $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$.
- $\lim_{x\to\infty}\left(1+\frac{t}{x}\right)^x = e^t$.
- $\lim_{x\to 0}\frac{e^x-1}{x} = 1$.

Proof.

- By $\lim_{y\to\infty}(1+\frac{1}{y})^y = e$ and a change of variables $y = \frac{1}{x}$, note that $\frac{1}{x} > 0$, $\lim_{x\to 0^+}(1+x)^{\frac{1}{x}} = e$.
  We have $\lim_{x\to 0-}(1+x)^{\frac{1}{x}} = e$ as well. So we have checked both the right and left limits.
- As $\log y$ is continuous at $y = e$,

$$\lim_{x\to 0}\frac{\log(1+x)}{x} = \lim_{x\to 0}\log(1+x)^{\frac{1}{x}} = \log\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \log e = 1,$$

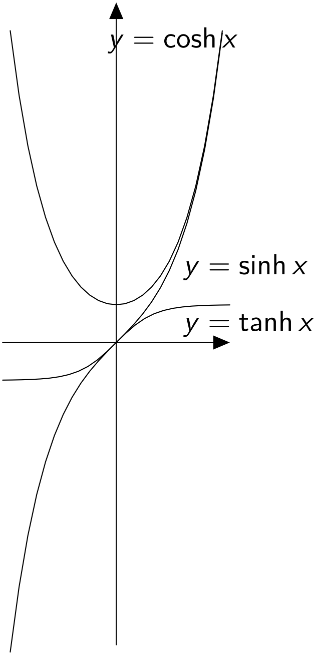
  where we used $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$.
- Note that $\lim_{x\to\infty}\left(1+\frac{t}{x}\right)^x = \lim_{x\to\infty}\left(\left(1+\frac{t}{x}\right)^{\frac{x}{t}}\right)^t = e^t$, where we used the continuity of $g(a) = a^t$.
- With $y = e^x - 1$, we have $\log(y+1) = x$ and $\lim_{x\to 0}\frac{e^x-1}{x} = \lim_{y\to 0}\frac{y}{\log(1+y)} = 1$.

□

The hyperbolic functions

Definition

- $\sinh x = \frac{e^x - e^{-x}}{2}$
- $\cosh x = \frac{e^x + e^{-x}}{2}$
- $\tanh x = \frac{\sinh x}{\cosh x}$



$y = \cosh x$
$y = \sinh x$
$y = \tanh x$

The hyperbolic functions

Lemma

- $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.
- $\sinh(x+y) = \cosh x \sinh y + \sinh x \cosh y$.
- $(\cosh x)^2 - (\sinh x)^2 = 1$.

Proof.

- $\cosh x \cosh y + \sinh x \sinh y = \frac{1}{4}(e^x + e^{-x})(e^y + e^{-y}) + \frac{1}{4}(e^x - e^{-x})(e^y - e^{-y}) = \frac{1}{4}(2e^{x+y} + 2e^{-x-y}) = \cosh(x+y)$.
- analogous.
- analogous.

□

Lemma

- $\sinh^{-1}(x) = \log(x + \sqrt{x^2 + 1})$.
- $\cosh^{-1}(x) = \log(x + \sqrt{x^2 - 1})$ *for* $x > 1$.

Proof.

-
$$\sinh(\log(x + \sqrt{x^2 + y})) = \frac{1}{2}\left((x + \sqrt{x^2 + 1}) - \frac{1}{x + \sqrt{x^2 + 1}}\right)$$
$$= \frac{1}{2}\frac{(x + \sqrt{x^2 + 1})^2 - 1}{x + \sqrt{x^2 + 1}}$$
$$= \frac{1}{2}\frac{x^2 + 2x\sqrt{x^2 + 1} + x^2 + 1 - 1}{x + \sqrt{x^2 + 1}} = x.$$
- analogous.

□

Definition

1. $\mathrm{Arcsinh}\, x = \sinh^{-1}(x) = \log(x + \sqrt{x^2 + 1})$.
2. $\mathrm{Arccosh}\, x = \cosh^{-1}(x) = \log(x + \sqrt{x^2 - 1})$ for $x > 1$.

Exercises

- Compute $\lim_{n\to 0}(1+\frac{1}{n})^{n^2}$.
- Compute $\lim_{x\to 0}\frac{\log_a(1+x)}{x}$.
- Compute $\lim_{x\to 0}\frac{a^x-1}{x}$.
- Prove $\lim_{x\to 0}\frac{\sinh x}{x}=1$.
- Prove $\lim_{x\to\infty}\tanh x=1$.
- Prove $\lim_{x\to 0}\frac{\sinh x}{e^x-1}=1$.