Mathematical Analysis I: Lecture 14

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Some properties of exponential functions

For $0 < a$ we have introduced a^x for real numbers x, by taking $x_n \in \mathbb{Q}, x_n \to x$ and $a^x := \lim_{n\to\infty} a^{x_n}$.

Theorem

We have the following.

- *For $a > 1$, $f(x) = a^x$ is monotonically increasing and continuous.*
- $a^x a^y = a^{x+y}$.
- $(a^x)^y = a^{xy}$.

Proof.

- Let $x < y$. Then we take sequences $x_n \to x, y_n \to y$, where $x_n, y_n \in \mathbb{Q}$. Then for sufficiently large n we have $x_n < z_1 < z_2 < y_n$ where $z_1, z_2 \in \mathbb{Q}$, and therefore, $a^x \leq a^{z_1} < a^{z_2} \leq a^y$.
  As for continuity, let us take $x, x_n \in \mathbb{R}$ and $x_n \to x$. Then there is $y_n \in \mathbb{Q}$ such that $|a^{x_n} - a^{y_n}| < \frac{1}{n}$ and $|x_n - y_n| < \frac{1}{n}$. Then $y_n \to x$ as well, hence $a^{y_n} \to a^x$, while $a^{y_n} - \frac{1}{n} < a^{x_n} < a^{y_n} + \frac{1}{n}$, therefore, $a^{x_n} \to a^x$.
- Take sequences $x_n \to x, y_n \to y$, $x_n, y_n \in \mathbb{Q}$. We have $a^{x_n} a^{y_n} = a^{x_n + y_n}$, and $x_n + y_n \to x + y$, therefore, $a^x a^y = a^{x+y}$.
- Take sequences $x_n \to x, y_n \to y$, $x_n, y_n \in \mathbb{Q}$. For fixed m, we have $(a^{x_n})^{y_m} \to (a^x)^{y_m}$ and this is equal to $a^{x_n y_m} \to a^{x y_m}$. Now we take the limit $m \to \infty$ and obtain $(a^x)^y = a^{xy}$ by continuity.

□

Napier's number

Let us introduce Napier's number. We put

$$e_n = \left(1 + \frac{1}{n}\right)^n, \qquad E_n = \left(1 + \frac{1}{n}\right)^{n+1} = \left(1 + \frac{1}{n}\right) e_n$$

Lemma

For $x \geq -1$, we have $(1+x)^n \geq 1 + nx$ for all n.

Proof.

By induction. With $n = 0, 1$, we have $(1+x)^0 = 1 = 1$ and $1 + x = 1 + x$. Assuming that this holds for n, we expand it and use $1 + 2n + nx \geq 0$:

$$\begin{aligned}(1+x)^{n+2} &= (1+x)^n(1+x)^2 \geq (1+nx)(1+x)^2 \\ &= 1 + nx + 2x + x^2 + 2nx^2 + nx^3 \\ &= 1 + (n+2)x + x^2(1 + 2n + nx) \geq 1 + (n+2)x.\end{aligned}$$

This completes the induction for even and odd numbers. □

Theorem

e_n and E_n converge to the same number e.

Proof.

The proof of this theorem requires several steps.

- We have $1 < e_n < E_n$. Indeed, $1 < 1 + \frac{1}{n}$, and this follows easily.
- e_n is monotinically increasing, that is, $e_n < e_{n+1}$. Indeed,

$$
\begin{aligned}
\frac{e_n}{e_{n-1}} &= \frac{(1+\frac{1}{n})^n}{(1+\frac{1}{n-1})^{n-1}} = \frac{(1+\frac{1}{n})^n}{(\frac{n}{n-1})^{n-1}} = (1+\tfrac{1}{n})^n \cdot (\tfrac{n-1}{n})^{n-1} \\
&= \frac{(1+\frac{1}{n})^n \cdot (\frac{n-1}{n})^n}{\frac{n-1}{n}} = \frac{(1+\frac{1}{n})^n \cdot (1-\frac{1}{n})^n}{\frac{n-1}{n}} \\
&= \frac{(1-\frac{1}{n^2})^n}{\frac{n-1}{n}} \geq \frac{1-\frac{1}{n}}{1-\frac{1}{n}} = 1.
\end{aligned}
$$

Proof.

- Similarly E_n is monotonically decreasing. Indeed.

$$\frac{E_n}{E_{n-1}} = \frac{(1+\frac{1}{n})^{n+1}}{(1+\frac{1}{n-1})^n} = \frac{1+\frac{1}{n}}{(\frac{n}{n-1})^n(\frac{n}{n+1})^n} = \frac{1+\frac{1}{n}}{(\frac{n^2}{n^2-1})^n}$$
$$= \frac{1+\frac{1}{n}}{(1+\frac{1}{n^2-1})^n} \leq \frac{1+\frac{1}{n}}{1+\frac{n}{n^2-1}} < \frac{1+\frac{1}{n}}{1+\frac{1}{n}} = 1.$$

- Now we have that $\{e_n\}$ and $\{E_n\}$ are convergent. Note also that $E_n - e_n = e_n(1+\frac{1}{n}-1) = e_n \cdot \frac{1}{n} \to 0$, because e_n is bounded, say by M, and $\frac{1}{n} \to 0$, therefore, $E_n - e_n \leq \frac{M}{n} \to 0$.

$\square$

Napier's number

We call this limit e, the **Napier's number** (sometimes **Euler's number**). The function e^x plays a special role in analysis, as we will see in the coming lectures.

Logarithm

Let $a > 0, a \neq 1$. We have defined the exponential function $f(x) = a^x$, and we have seen that it is continuous, monotonically increasing if $a > 1$. If $0 < a < 1$, it is monotonically decreasing.
Let $a > 1$. We know that a^n diverges, and hence $a^{-n} \to 0$. By the intermediate value theorem, we see that the range of a^x is $\mathbb{R}_+$. Now we can define the inverse function (everything is analogous for $0 < a < 1$).

Logarithm

Definition

The **logarithm base** a **of** x $\log_a x$ is the inverse function $f(y) = a^y$: $\log_a : \mathbb{R}_+ \longmapsto \mathbb{R}$ and it holds that

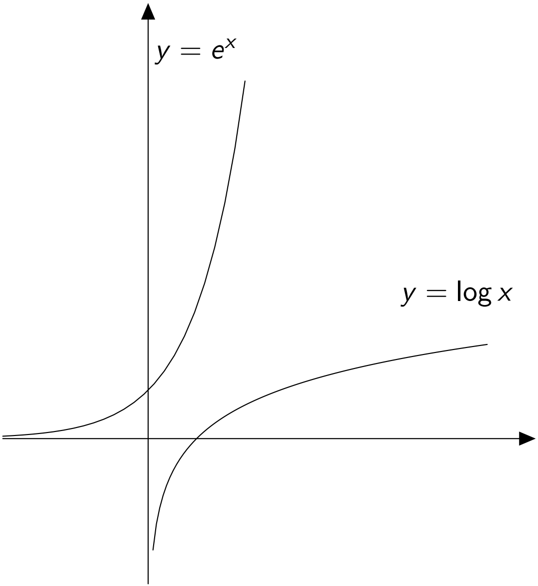
$$\log_a a^x = x = a^{\log_a x}.$$

We denote $\log x = \log_e x = \ln x$.

Example

$\log_2 8 = 3, \log_9 3 = \frac{1}{2}$.

Logarithm



Logarithm

We say that $\lim_{x\to\infty} f(x) = \infty$ if for each $Y > 0$ there is $X > 0$ such that if $x > X$ then $f(x) > Y$. Similarly, we define $\lim_{x\to\pm\infty} f(x) = \pm\infty$.

Theorem

Let $a, b > 0$, $a \neq 1 \neq b$, $x, y > 0$, $t \in \mathbb{R}$. Then

1. $\log_a a = 1, \log_a 1 = 0$.
2. $\log_a(xy) = \log_a x + \log_a y$.
3. $\log_a(x^t) = t \log_a x$.
4. $\log_{a^{-1}} x = -\log_a x$.
5. $\log_a x = \log_a b \cdot \log_b x$.
6. *Let $a > 1$. Then $f(x) = \log_a x$ is monotonically increasing and continuous. $\log_a x > 0$ if and only if $x > 1$.*
7. *Let $a > 1, \alpha > 0$. Then $\lim_{x\to+\infty} \frac{x^\alpha}{\log_a x} = +\infty$.*

Logarithm

Proof.

1. $a^1 = a, a^0 = 1$. Therefore, $\log_a a = 1, \log_a 1 = 0$.
2. $a^{\log_a x + \log_a y} = a^{\log_a x} a^{\log_a y} = xy$. Therefore, $\log_a(xy) = \log_a x + \log_a y$.
3. $a^{t \log_a x} = (a^{\log_a x})^t = x^t$.
4. $(1/a)^{-\log_a x} = 1/a^{-\log_a x} = 1/(a^{\log_a x})^{-1} = 1/x^{-1} = x$.
5. $a^{\log_a b \cdot \log_b x} = (a^{\log_a b})^{\log_b x} = b^{\log_b x} = x$.

Logarithm

Proof.

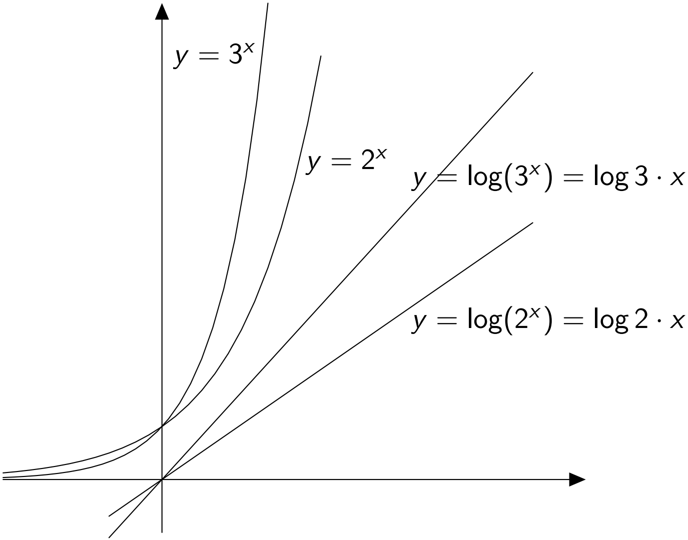
1. This follows from the monotonicity and continuity of a^x.
2. First we show that $\lim_{n\to\infty} \frac{(a^{n-1})^\alpha}{\log_a(a^n)} = \infty$. This is straightforward because $\frac{(a^{n-1})^\alpha}{\log_a(a^n)} = \frac{a^{(n-1)\alpha}}{n} \to \infty$. To show the given limit, we take for $y > 0$ $n \in \mathbb{N}$ such that $n - 1 \leq y < n$. Then $\frac{(a^y)^\alpha}{\log_a a^y} > \frac{(a^{n-1})^\alpha}{\log_a(a^n)}$, and hence the left-hand side grows as y grows. That is, $\lim_{y\to\infty} \frac{(a^y)^\alpha}{\log_a a^y} = \infty$. Finally, recall that $x = a^y$ is monotonic, and x grows infinitely as y grows. That is, given $Z > 0$, there is $Y > 0$ such that $\frac{(a^y)^\alpha}{\log_a a^y} > Z$ for $y > Y$, which implies that $\frac{x^\alpha}{\log_a x} > Z$ for $x > a^Y$. This means that $\lim_{x\to\infty} \frac{x^\alpha}{\log_a x} = \infty$.

$\square$

Logarithm

Logarithm is extremely useful in natural science. When we have a data which grows exponentially, we can take the log of the value and plot it to a plane, then they lie on a straight line. The exponent can be read from the slope of the line (this is called the logarithmic scale). In that case, the logarithm base 10 is often used.

Logarithm



Logarithm

When $y = x^p$, then we can consider $z = \log y, w = \log x$, hence $e^z = y, e^w = y$. We have $e^z = y = x^p = (e^w)^p = e^{wp}$. By taking log of both side, we obtain $z = pw$. That is, by the log-log plot, a power relation $y = x^p$ is translated into a linear relation $z = pw$.

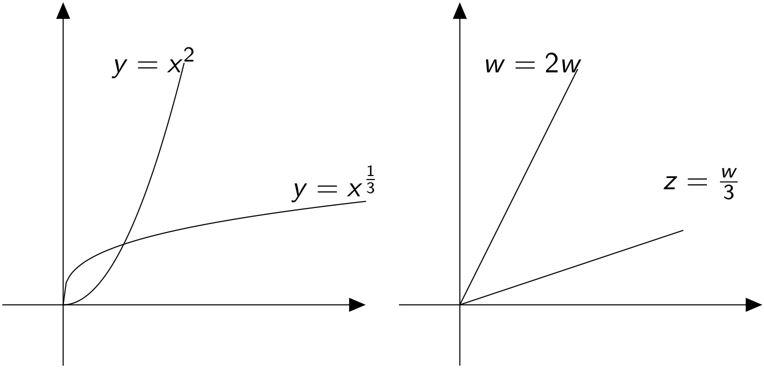


Figure: The log-log plot of the relation $y = x^p$.

Exercises

- Compute $\log_3(81), \log_{81} 3$.
- Compute $(1 + \frac{1}{3})^3$.
- If $y = Ce^{ax}$, what is the relation between $z = \log y$ and x?
- If $y = Cx^p$, what is the relation between $z = \log y$ and $w = \log x$?
- Calculate the integer part of $\log_{10}(232720)$.
- Calculate the integer part of $\log_2(13567)$.