Mathematical Analysis I: Lecture 13

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Exponential functions

For $a > 0$ and $p, q \in \mathbb{N}$, we have defined $a^{\frac{p}{q}}$. Then the natural question arises whether a^x can be defined for real numbers x.
For a fixed $a > 0$, we can consider $f(x) = a^x$ as a function defined on the set of rational numbers $\mathbb{Q}$.

Exponential functions

Lemma

We have the following.

- *For* $p, q, r, s \in \mathbb{N}$*, we have* $a^{\frac{p}{q}} a^{\frac{r}{s}} = a^{\frac{p}{q}+\frac{r}{s}}$.
- *For* $p, q, r, s \in \mathbb{N}$*, we have* $(a^{\frac{p}{q}})^{\frac{r}{s}} = a^{\frac{pr}{qs}}$.
- *If* $a > 1$*, then* $f(x) = a^x$ *is monotonically increasing (as a function on* $\mathbb{Q}$*).*
- *If* $0 < a < 1$*, then* $f(x) = a^x$ *is monotonically decreasing.*

Proof.

- Recall that we have $a^{\frac{p}{q}} = a^{\frac{ps}{qs}}$ and $a^{\frac{r}{s}} = a^{\frac{qr}{qs}}$, and hence

$$a^{\frac{p}{q}} a^{\frac{r}{s}} = a^{\frac{ps}{qs}} a^{\frac{qr}{qs}} = (a^{\frac{1}{qs}})^{ps} (a^{\frac{1}{qs}})^{qr} = (a^{\frac{1}{qs}})^{ps+qr} = a^{\frac{ps+qr}{qs}} = a^{\frac{p}{q}+\frac{r}{s}}.$$

- We will prove this as an exercise.
- Let us take $a > 1$. First, for any $q \in \mathbb{N}$, $a^{\frac{1}{q}} > 1$, indeed, if $a^{\frac{1}{q}} \le 1$, we would have $a = (a^{\frac{1}{q}})^q \le 1$, contradiction.
  If $x_1, x_2 \in \mathbb{Q}$ and $x_1 < x_2$, we may assume that $x_1 = \frac{p}{q}, x_2 = \frac{r}{q}$ and $p < r$. Then

$$a^{x_1} = a^{\frac{p}{q}} = (a^{\frac{1}{q}})^p < (a^{\frac{1}{q}})^r = a^{\frac{r}{q}} = a^{x_2}.$$

- The case $0 < a < 1$ is similar.

$\square$

Exponential functions

We would like to define a^x by $\lim_{n\to\infty} a^{x_n}$, where $x_n \in \mathbb{Q}$ and $x_n \to x \in \mathbb{R}$. For this purpose, we need some properties of sequences.

Lemma

If $a_n \leq b_n$ and $a_n \to L, b_n \to M$, then $L \leq M$.

Proof.

Consider $b_n - a_n \geq 0$. We have $b_n - a_n \to M - L \geq 0$, hence $M \geq L$. □

Exponential functions

We write $a_n \to \infty$ if for any $x \in \mathbb{R}$ there is N such that $a_n > x$ for $n > N$.

Theorem

Let $a_n \le b_n \le c_n$ be three sequences. If $a_n \to L$ and $c_n \to L$, then $b_n \to L$. Similarly, if $a_n \to \infty$, then also $b_n \to \infty$.

Proof.

For a given $\epsilon > 0$, we take N such that for $n > N$ it holds that $|a_n - L| < \epsilon$ and $|c_n - L| < \epsilon$. For a fixed $n > N$, this means that $L - \epsilon < a_n \le b_n \le a_n < L + \epsilon$, and hence $|b_n - L| < \epsilon$. This means that $b_n \to L$.

If $a_n \to \infty$, then for a given x there is N such that $x < a_n \le b_n$, hence $b_n \to \infty$. □

For a statement like "there is N such that for $n > N$..." we say simply that "for sufficiently large n...".

Theorem

We have the following.

- *For $a > 1, p \in \mathbb{N}$, we have $\frac{a^n}{n^p}$ diverges.*
- *It holds that $n^{\frac{1}{n}} \to 1$.*
- *For $a > 1$, we have $a^{\frac{1}{n}} \to 1$.*

Proof.

- Let us consider first $p = 1$. Then, writing $a = 1 + y$ with $y > 0$, we have, for $n \geq 2$,

$$a^n = (1+y)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k y^{n-k} > 1 + \frac{n(n-1)}{2} y^2,$$

and hence $\frac{a^n}{n} > \frac{(n-1)y^2}{2}$. As $\frac{(n-1)y^2}{2} \to \infty$, so does it hold $\frac{a^n}{n} \to \infty$. For a general $p \in \mathbb{N}$, we take $a^{\frac{1}{p}}$, then $1 < a^{\frac{1}{p}}$ and $\frac{a^{\frac{n}{p}}}{n} \to \infty$, hence $\frac{a^n}{n^p} = \left(\frac{a^{\frac{n}{p}}}{n}\right)^p \to \infty$.

Proof.

- Let $\epsilon > 0$. We need prove that $n^{\frac{1}{n}} < 1 + \epsilon$ for sufficiently large n. Equivalently, $n < (1 + \epsilon)^n$. This follows from the previous point that $\frac{(1+\epsilon)^n}{n} \to \infty$, in particular, $\frac{(1+\epsilon)^n}{n} > 1$ for sufficiently large n.
- $1 < a^{\frac{1}{n}} < n^{\frac{1}{n}}$ for $a < n$, therefore the claim follows from the previous Theorem.

$\square$

Cauchy sequences

Definition

A sequence a_n is said to be a **Cauchy sequence** if for each $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $|a_m - a_n| < \epsilon$ for $m, n > N$.

Differently from the convergence to a number L, this says that two elements in the sequence are close to each other for large enough m, n.

Lemma

A sequence a_n is convergent if and only if it is a Cauchy sequence.

Proof.

If $a_n \to L$, then for any $\epsilon > 0$ we can take N such that $|a_n - L| < \frac{\epsilon}{2}$ for $n > N$, therefore, if $n, m > N$, then $|a_m - L| < \frac{\epsilon}{2}$ as well and hence $|a_m - a_n| \leq |a_m - L| + |L - a_n| < \epsilon$.

Proof.

Conversely, if a_n is Cauchy, then it is bounded. Indeed, we take N such that $|a_m - a_{N+1}| < 1$, then this means that $|a_m| < |a_{N+1}| + 1$. Then we can take the largest number among $|a_1|, \cdots, |a_N|, |a_{N+1}| + 1$ as a bound. Next, we consider the sequence

$$b_n = \inf\{a_k : k \geq n\}.$$

This is well-defined because $\{a_k : k \geq n\}$ is bounded. Furthermore, this sequence is increasing because $\{a_k : k \geq n+1\} \supset \{a_k : \kappa \geq n\}$. Therefore, b_n converges to some number L. Similarly, with $c_n = \sup\{a_k : k \geq n\}$, this is bounded and decreasing, hence converges to M.
Note that $b_n \leq a_n \leq c_n$, therefore, $L \leq M$. Actually, we have $L = M$. Indeed, for given $\epsilon > 0$, we can find sufficiently large ℓ, m, n such that $|c_n - M| < \frac{\epsilon}{5}, |a_\ell - c_n| < \frac{\epsilon}{5}, |b_n - L| < \frac{\epsilon}{5}, |a_m - b_n| < \frac{\epsilon}{5}$ and $|a_\ell - a_m| < \frac{\epsilon}{5}$. This implies that $|M - L| < \epsilon$ for arbitrary $\epsilon > 0$, hence it must hold $M = L$. Now, as $b_n, c_n \to L = M$ and $b_n \leq a_n \leq c_n$, we have $a_n \to L$ by the previous Theorem. □

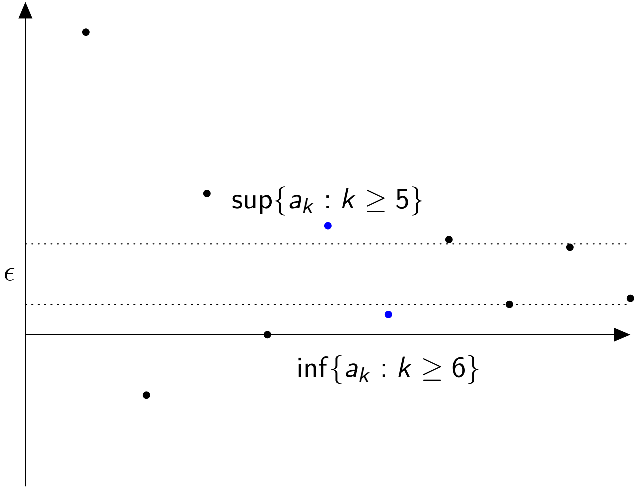


Figure: A Cauchy sequence.

Exponential functions

Finally, we can define a^x for all real number x.

Theorem

Let $a > 0, x_n \in \mathbb{Q}, x_n \to x$. Then a^{x_n} converges. If $y_n \in \mathbb{Q}, y_n \to x$, then $\lim_{n\to\infty} a^{x_n} = \lim_{n\to\infty} a^{y_n}$.

Proof.

Note that $\{x_n\}$ is bounded, hence $\{a^{x_n}\}$ is bounded as well, say by M, because the exponential function on $\mathbb{Q}$ is monotonic. We show that a^{x_n} is convergent. To see this, it is enought to see that a^{x_n} is Cauchy by the previous Lemma.

Proof.

For a given $\epsilon > 0$, we take $\delta > 0$ such that $|a^z - 1| < \frac{\epsilon}{M}$ for $0 < z < \delta$. For sufficiently large m, n, we have $|x_m - x_n| < \delta$ and in that case,

$$|a^{x_m} - a^{x_n}| = |a^{x_m}||1 - a^{x_n - x_m}| \leq M|1 - a^{x_n - x_m}| < M\frac{\epsilon}{M} = \epsilon.$$

This means that $\{a^{x_n}\}$ is a Cauchy sequence, and hence it converges to a certain real number, which we call a^x.

If $\{y_n\}$ is another sequence converging to x, then we can consider a further new sequence $x_1, y_1, x_2, y_2, \cdots$, and this converges to some number. But the subsequence $\{x_n\}$ converges to a^x, and hence the whole sequence and hence $\{y_n\}$ must converge to a^x as well. $\square$

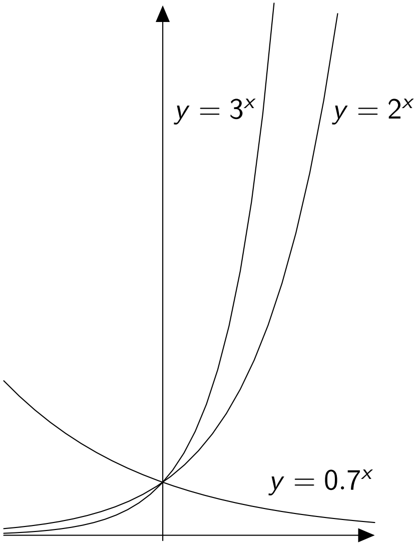
Exponential functions

As we said in the proof, for an arbitrary real number $x \in \mathbb{R}$, we define the **exponential function** by

$$a^x := \lim_{n\to\infty} a^{x_n}, \text{ where } x_n \in \mathbb{Q}, x_n \to x.$$

The exponential functions appear in various natural phenomena. It happens typically when we consider a collection of objects that increase or decrease independently (such as colonies of bacteria, radioactive nuclei, and so on).

Exponential functions



Exercises

- Prove that for $p, q, r, s \in \mathbb{N}$, we have $(a^{\frac{p}{q}})^{\frac{r}{s}} = a^{\frac{pr}{qs}}$.
- Let $a_n = \frac{(-1)^n}{n}$. Determine $\sup\{a_k : k \geq n\}$ and $\inf\{a_k : k \geq n\}$.
- Compute 2^x for $x = 1, 2, 3, 4, \frac{1}{2}, \frac{3}{2}$.
- Compute $(\frac{1}{9})^x$ for $x = 1, 2, 3, \frac{1}{2}, \frac{3}{2}$.
- Imagine that there is a pond and the leaves of lotus doubles each day. If the pond is completely filled on day 100, when is the pond half filled?
- Rewrite 2^{ax} in a form b^x for some $b > 0$.