Mathematical Analysis I: Lecture 12

Lecturer: Yoh Tanimoto

09/10/2020

Start recording...

Exercises

Let $a_n = \frac{1}{\sqrt{\sqrt{n}}}$ and $\epsilon = 0.01$. Find N such that for $n > N$ it holds $|a_n| < \epsilon$.

Let $a_n = \frac{1}{\sqrt{\sqrt{n}}}$ and $\epsilon = 0.01$. Find N such that for $n > N$ it holds $|a_n| < \epsilon$.

Solution. Note that $\sqrt{\sqrt{100000000^{-1}}} = \sqrt{\sqrt{0.00000001}} = 0.01$, hence if $n > 100000000$, then $\frac{1}{\sqrt{\sqrt{n}}} < \frac{1}{\sqrt{\sqrt{100000000}}} = 0.01$. We can take $N = 100000000$.

Let $a_n = \frac{1}{2^n}$ and $\epsilon = 0.00001$. Find N such that for $n > N$ it holds $|a_n| < \epsilon$.

Let $a_n = \frac{1}{2^n}$ and $\epsilon = 0.00001$. Find N such that for $n > N$ it holds $|a_n| < \epsilon$.

Solution. Note that $2^{17} = 131072 > 100000$, hence $\frac{1}{2^{17}} < \frac{1}{100000} = 0.00001$. As $\frac{1}{2^n} > \frac{1}{2^{n+1}}$, we can take $N = 17$.

Show that a constant sequence $a_n = C \in \mathbb{R}$ is convergent.

Show that a constant sequence $a_n = C \in \mathbb{R}$ is convergent.
Solution. For any given $\epsilon > 0$ we can take $N = 1$ and then for any $n > 1$ we have $|a_n - C| = |C - C| = 0 < \epsilon$.

Tell whether $\{a_n\}$ converges, and if it does, compute the limit $a_n = \frac{1}{1+\frac{1}{n}}$.

Tell whether $\{a_n\}$ converges, and if it does, compute the limit $a_n = \frac{1}{1+\frac{1}{n}}$.

Solution. $\frac{1}{n}$ converges to 0, and $1 + \frac{1}{n}$ converges to 1 (sum), and $\frac{1}{1+\frac{1}{n}}$ converges to $\frac{1}{1} = 1$ (quotient with nonzero denominator).

Tell whether $\{a_n\}$ converges, and if it does, compute the limit $a_n = \frac{n}{1+n}$.

Tell whether $\{a_n\}$ converges, and if it does, compute the limit $a_n = \frac{n}{1+n}$.
Solution. Note that $\frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}$, hence this converges to 1 by the previous problem.

Tell whether $\{a_n\}$ converges, and if it does, compute the limit
$a_n = \frac{n^3+n^2+4}{n^3+100}$.

Tell whether $\{a_n\}$ converges, and if it does, compute the limit $a_n = \frac{n^3+n^2+4}{n^3+100}$.

Solution. Note that $\frac{n^3+n^2+4}{n^3+100} = \frac{1+\frac{1}{n}+\frac{4}{n^2}}{1+\frac{100}{n^3}}$. The numerator tends to 1 and the denominator tends to 1 as well, therefore, $a_n \to 1$.

Let $x = 0.12341234\cdots$. Represent x as a rational number.

Let $x = 0.12341234\cdots$. Represent x as a rational number.
Solution. x is approximated by

$$\begin{aligned}0.1 + 0.02 + 0.003 + 0.0004 + \cdots &= \sum_{k=1}^{n} 1234 \cdot 10000^{-k} \\ &= \frac{1234(1 - 10000^{-n}}{1 - 10000} \\ &\to \frac{1234}{10000 - 1} = \frac{1234}{9999}.\end{aligned}$$

Compute $\lim_{x \to 2} x^2$.

Compute $\lim_{x\to 2} x^2$.
Solution. We have seen that $f(x) = x$ is continuous, therefore, $\lim_{x\to 2} x = 2$ and with $g(x) = x \cdot x$ we have $\lim_{x\to 2} x^2 = 2 \cdot 2 = 4$.

Compute $\lim_{x \to 1} \frac{x+2}{x-3}$.

Compute $\lim_{x\to 1} \frac{x+2}{x-3}$.
Solution. It is easy to see that $f(x) = x + 2$ and $g(x) = x - 3$ are continuous, therefore, the quotient $\frac{x+2}{x-3}$ is continuous as long as $x \neq 3$. That is, $\lim_{x\to 1} \frac{x+2}{x-3} = \frac{\lim_{x\to 1} x+2}{\lim_{x} \to 1 x-3} = \frac{3}{-2} = -\frac{3}{2}$.

Compute $\lim_{x \to -1} \frac{x^2+3x+2}{x^2-1}$.

Compute $\lim_{x \to -1} \frac{x^2+3x+2}{x^2-1}$.

Solution. As it is written, the denominator tends to 0 as $x \to -1$. But actually we have $\frac{x^2+3x+2}{x^2-1} = \frac{(x+2)(x+1)}{(x-1)(x+1)} = \frac{x+2}{x-1}$ for $x \neq -1$. Therefore,

$$\lim_{x \to -1} \frac{x^2+3x+2}{x^2-1} = \lim_{x \to -1} \frac{x+2}{x-1} = \frac{1}{-2} = -\frac{1}{2}.$$

Let $f(x) = \begin{cases} x^2 & \text{if } x \geq 1 \\ 0 & \text{if } x \leq 1 \end{cases}$. Is f continuous or not? If not, where is it not continuous?

Let $f(x) = \begin{cases} x^2 & \text{if } x \geq 1 \\ 0 & \text{if } x \leq 1 \end{cases}$. Is f continuous or not? If not, where is it not continuous?

Solution. We know that x^2 and 0 are continuous for $x > 1$ and $x < 1$, respectively. The problem is at $x = 1$. If $x_n > 1, x_n \to 1$, then $f(x_n) = x_n^2 \to 1$, but if $x_n < 1, x_n \to 1$, then $f(x_n) = 0 \to 0$, and they do not conincide. Hence f is not continuous at $x = 1$.

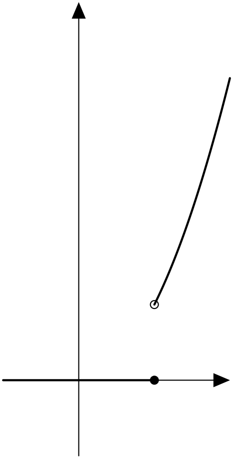

Figure: The graphs of $f(x) = \begin{cases} x^2 & \text{if } x \geq 1 \\ 0 & \text{if } x \leq 1 \end{cases}$.

Let $f(x) = \begin{cases} \frac{x^2+3x+2}{x^2-1} & \text{for } x \neq 1, -1 \\ -\frac{1}{2} & \text{for } x = -1 \end{cases}$, defined on $\mathbb{R} \setminus \{1\}$. Is f continuous or not? If not, where is it not continuous?

Let $f(x) = \begin{cases} \frac{x^2+3x+2}{x^2-1} & \text{for } x \neq 1, -1 \\ -\frac{1}{2} & \text{for } x = -1 \end{cases}$, defined on $\mathbb{R} \setminus \{1\}$. Is f continuous or not? If not, where is it not continuous?

Solution. As we saw before, $\frac{x^2+3x+2}{x^2-1} = \frac{x+2}{x-1}$ and $\lim_{x\to -1} \frac{x^2+3x+2}{x^2-1} = -\frac{1}{2}$. As $f(-1) = -\frac{1}{2}$ by definition, f is continuous at $x = -1$. It is also continuous at $x \neq 1$. Therefore, it is continuous on $\mathbb{R} \setminus \{1\}$ (not defined at $x = 1$).

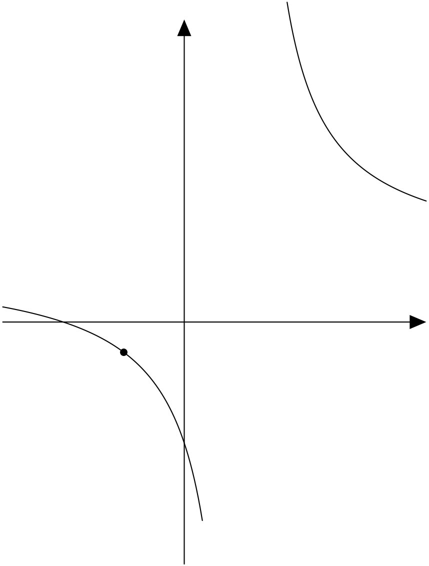

Figure: The graph of $f(x) = \frac{x^2+3x+2}{x^2-1}$.

Let $f(x) = x^4 + 3x^3 - x - 2$. Show that the equation $f(x) = 0$ has at least two solutions.

Let $f(x) = x^4 + 3x^3 - x - 2$. Show that the equation $f(x) = 0$ has at least two solutions.

Solution. Note that $f(0) = -2, f(1) = 1$. Hence by the intermediate value theorem there is $x_1 \in (-2, 1)$ such that $f(x_1) = 0$. Similarly, $f(0) = -2, f(-3) = 1$. Hence by the intermediate value theorem there is $x_2 \in (-3, 0)$ such that $f(x_2) = 0$.

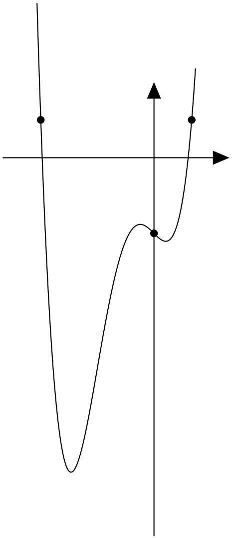

Figure: The graphs of $f(x) = x^4 + 3x^3 - x - 2$.

Compute $\lim_{x \to 1} \sqrt{x + 3\sqrt{x}}$.

Compute $\lim_{x\to 1}\sqrt{x+3\sqrt{x}}$.

Solution. We know that $\sqrt{x}=x^{\frac{1}{2}}$ is continuous (on $\mathbb{R}_+\cup\{0\}$), hence $\lim_{x\to 1}\sqrt{x}=1$. Further $x+3\sqrt{x}$ is continuous and $\lim_{x\to 1}x+3\sqrt{x}=4$. Finally $\lim_{x\to 1}\sqrt{x+3\sqrt{x}}$ is continuous (on $\mathbb{R}_+\cup\{0\}$) and $\lim_{x\to 1}\sqrt{x+3\sqrt{x}}=\sqrt{4}=2$.

Show that $a^{\frac{1}{n}} b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$ for $a, b \geq 0$.

Show that $a^{\frac{1}{n}} b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$ for $a, b \geq 0$.
Solution. Note that $(a^{\frac{1}{n}} b^{\frac{1}{n}})^n = (a^{\frac{1}{n}})^n (b^{\frac{1}{n}})^n = ab$, hence we can take the n-th root of both sides.

Compute $\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x^2}$.

Compute $\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x^2}$.

Solution. At first sight, it would yield $\frac{0}{0}$. However, for $x \neq 0$, we have

$$\begin{aligned}\frac{1-\sqrt{1-x^2}}{x^2} &= \frac{(1-\sqrt{1-x^2})(1+\sqrt{1-x^2})}{x^2(1+\sqrt{1-x^2})} \\ &= \frac{1-(1-x^2)}{x^2(1+\sqrt{1-x^2})} \\ &= \frac{1}{1+\sqrt{1-x^2}}\end{aligned}$$

Therefore, $\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x^2} = \lim_{x\to 0} \frac{1}{1+\sqrt{1-x^2}} = \frac{1}{2}$.

Compute $\lim_{x\to 0} \frac{\sqrt{1-x}-\sqrt{1+x}}{x}$.

Compute $\lim_{x\to 0} \frac{\sqrt{1-x}-\sqrt{1+x}}{x}$.
Solution.

$$
\begin{aligned}
\frac{\sqrt{1-x}-\sqrt{1+x}}{x} &= \frac{(\sqrt{1-x}-\sqrt{1+x})(\sqrt{1-x}+\sqrt{1+x})}{x(\sqrt{1-x}+\sqrt{1+x})} \\
&= \frac{(1-x)-(1+x)}{x(\sqrt{1-x}+\sqrt{1+x})} \\
&= \frac{-2}{\sqrt{1-x}+\sqrt{1+x}}
\end{aligned}
$$

Therefore, $\lim_{x\to 0} \frac{\sqrt{1-x}-\sqrt{1+x}}{x} = \lim_{x\to 0} \frac{-2}{\sqrt{1-x}+\sqrt{1+x}} = -1$.

Consider $f(x) = x^2$. For $\epsilon = 0.1$, find a δ which shows the continuity of f at $x = 1$.

Consider $f(x) = x^2$. For $\epsilon = 0.1$, find a δ which shows the continuity of f at $x = 1$.
Solution. Note that $(1 + y)^2 = 1 + 2y + y^2$. We need that $|2y + y^2| < 0.1$, and this is achieved with $|y| < 0.04$.

Consider $f(x) = x^{\frac{1}{3}}$. For $\epsilon = 0.1$, find a δ which shows the continuity of f at $x = 0$.

Consider $f(x) = x^{\frac{1}{3}}$. For $\epsilon = 0.1$, find a δ which shows the continuity of f at $x = 0$.

Solution. We need that $x^{\frac{1}{3}} < 0.1$, hence $x < 0.001$ (and $x \geq 0$).