Mathematical Analysis I: Lecture 11

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Start recording...

Tips

- Today: Apostol Vol 1, Chapter 3.7-14.
- Are proofs asked in the exam? **No to most students** (maybe yes to get "lode")
- But proof give you the idea of why certain things are correct, an occasion to undertand better several definitions (and assure us that correct things are correct).

Sequences and continuity of functions

Let f be a function defined on a certain domain S and $\{x_n\}$ a sequence in S. Then we can construct a new sequence by $\{f(x_n)\}$.

Theorem

Let f be a function defined on S. f is continuous at $a \in S$, that is, $\lim_{x\to a} f(x) = f(a)$ if and only if it holds that $f(x_n) \to f(a)$ for all sequences $\{x_n\}$ such that $x_n \to a, x_n \neq a$.

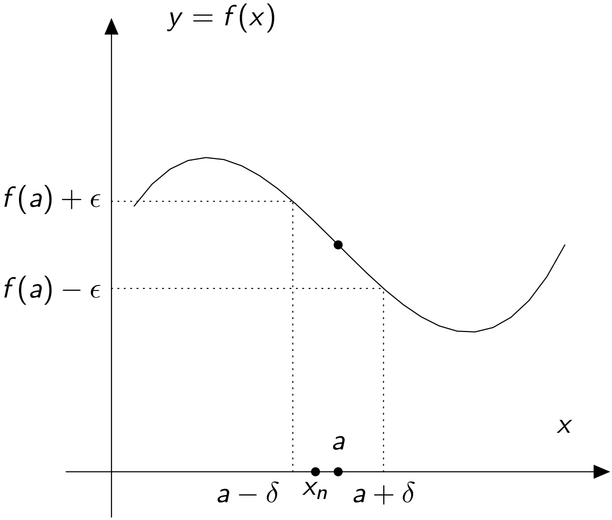
Proof.

Assume that $\lim_{x\to a} f(x) = f(a)$. Then, for each $\epsilon > 0$, there is $\delta > 0$ such that if $|x - a| < \delta$, then it holds that $|f(x) - f(a)| < \epsilon$. Let us take any sequence $\{x_n\}$ such that $x_n \to a$. This means that, for δ above, there is N such that $|x_n - a| < \delta$ for $n > N$. Then by the observation above, we have $|f(x_n) - f(a)| < \epsilon$. This shows that, for $n > N$, we have $|f(x_n) - f(a)| < \epsilon$. Therefore, for the given ϵ we found N such that $|f(x_n) - f(a)| < \epsilon$ for $n > N$. This means that $f(x_n) \to f(a)$.

Proof.

Conversely, assume that $f(x_n) \to f(a)$ for all sequences $\{x_n\}$ such that $x_n \to a, x_n \neq a$. To do a proof by contradiction, let us assue that there is $\epsilon > 0$ for which for all δ there is $x \in S, x \neq a$ such that $|x - a| < \delta$ but $|f(x) - f(a)| > \epsilon$. Let us take $\delta_n = \frac{1}{n}$. For each δ_n there is $x_n \in S$ such that $|x_n - a| < \frac{1}{n}, x \neq a$ but $|f(x_n) - f(a)| > \epsilon$. Then, it is clear that $x_n \to a$, but $f(x_n)$ is not converging to $f(a)$, which contradicts the assumption. Therefore, it must hold that $\lim_{x\to a} f(x) = f(a)$.

□



$y = f(x)$
$f(a) + \epsilon$
$f(a) - \epsilon$
x
a
$a - \delta$
x_n
$a + \delta$

Lemma

Let $x_n \to x$ and $x_n \leq a$. Then $x \leq a$.

Proof.

Assume the contrary, that is, $x > a$. Then there is N such that $|x_n - x| < \frac{x-a}{2}$, and $x_n - a = x_n - x + x - a > |x - a| - \frac{|x-a|}{2} = \frac{|x-a|}{2} > 0$, which contradicts $x_n \leq a$. Therefore, $x \leq a$. □

It also holds that, if $x_n \to x$ and $x_n \geq a$, then $x \geq a$.
We can show that, if $A \subset [a, b]$, then $\sup A \in [a, b]$: indeed, we can take a sequence $\{x_n\} \subset A$ such that $x_n \to \sup A$.

The intermediate value theorem

Theorem

Let f be a continuous function on a closed interval $[a, b]$. Assume that $f(a) < f(b)$. Then, for any value $c \in (f(a), f(b))$, there is $x \in (a, b)$ such that $c = f(x)$.

Proof.

Let $c \in (f(a), f(b))$, and we define $A = \{x \in [a, b] : f(x) < c\}$. A is bounded above, because it is contained in $[a, b]$, therefore, we can take $x = \sup A$, and $x \in [a, b]$ by the previous Lemma. By the property of sup, for each n, there is $x_n \in A$ such that $x - \frac{1}{n} < x_n$, hence $x_n \to x$. Since f is continuous, we have $f(x) = \lim_{n\to\infty} f(x_n)$.

Proof.

On the other hand, $x_n \in A$, hence $f(x_n) < c$ and hence $f(x) \leq c$ by the previous Lemma. $x \neq b$ because $f(b) > c \geq f(x)$. Therefore, we can take a sequence $x_n > x, x_n \to x$ in the interval $[x, b]$, and then $f(x_n) \geq c$ because $x_n \notin A$. By continuity of f, we have $f(x) = \lim_n f(x_n) \geq c$. Altogether, we have $f(x) = c$. □

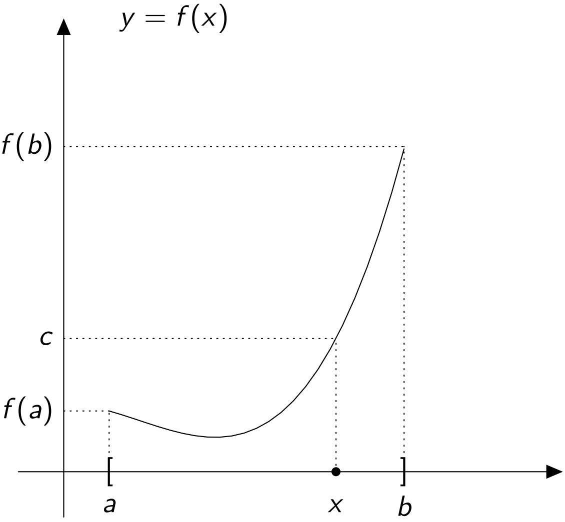


Figure: The intermediate value c is taken at x.

Composition and inverse functions

Let f, g be two functions, f defined on S and g defined on the image (range) of f: $f(S) = \{y \in \mathbb{R} : \text{ there is } x \in S, y = f(x)\}$. Recall that we can compose two functions: for $x \in S$, $g(f(x))$ gives a number, hence the correspondence $x \to g(f(x))$ is a function on S. We denote this **composed function** by $g \circ f$.

Theorem

In the situation above, if f and g are continuous, then $g \circ f$ is continuous as well.

Proof.

We take $a \in S$. Given $\epsilon > 0$, there is $\delta_1 > 0$ such that $|g(y) - g(f(a))| < \epsilon$ if $|y - f(a)| < \delta_1$, by continuity of g. For this δ_1, there is $\delta_2 > 0$ such that $|f(x) - f(a)| < \delta_1$ if $|x - a| < \delta_2$. Altogether, we have $|g(f(x)) - g(f(a))| < \epsilon$ if $|x - a| < \delta_2$, hence we have proved the continuity of $g \circ f$. □

Monotonic functions

Definition

Let f be a function on S. We say that f is **monotonically increasing (nondecreasing, decreasing, nonincreasing, respectively)** if for each $x_1, x_2 \in S, x_1 < x_2$ it holds that $f(x_1) < f(x_2)$ $(f(x_1) \leq f(x_2), f(x_1) > f(x_2), f(x_1) \geq f(x_2)$, respectively).

Example

(Non)examples of monotonic functions.

- $f(x) = x$ is monotonically increasing.
- $f(x) = x^2$ is not monotonically increasing on $\mathbb{R}$, but it is so on $\mathbb{R}_+$.
- $f(x) = \operatorname{sign} x$ is monotonically nondecreasing.

If a function f is monotonically increasing (or decreasing), it is injective: for $x_1 \neq x_2$, it holds that $f(x_1) \neq f(x_2)$. Therefore, we can consider its inverse function.

Inverse functions

Theorem

Let f be a monotonically increasing function on an interval $[a, b]$. Then the inverse function f^{-1} defined on $[f(a), f(b)]$ is monotonically increasing and continuous.

Proof.

Note that the domain of f^{-1} is $[f(a), f(b)]$ by the intermediate value theorem.
Let us first show that f^{-1} is monotonically increasing. For each $y_1 < y_2, y_1, y_2 \in [f(a), f(b)]$, there are $x_1, x_2 \in S$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$ by the intermediate value theorem and we have $x_1 < x_2$ by monotonicity of f. This means that $f^{-1}(y_1) = x_1 < x_2 = f^{-1}(y_2)$, that is, f^{-1} is monotonically increasing.

Proof.

Let $x_0 \in (a, b)$. For a given $\epsilon > 0$, we take δ as the smaller of $f(x_0 + \epsilon)$ and $f(x_0 - \epsilon)$ (if $x_0 \pm \epsilon$ are not in S, replace them by a or b). Then for any $y \in (f(x_0) - \delta, f(x_0) + \delta)$, we have $f^{-1}(y) \in S \cap (x_0 - \epsilon, x_0 + \epsilon)$ by monotonicity of f. This is the continuity of f^{-1}.

If $x_0 = a$ or b, then we only have to consider one side. □

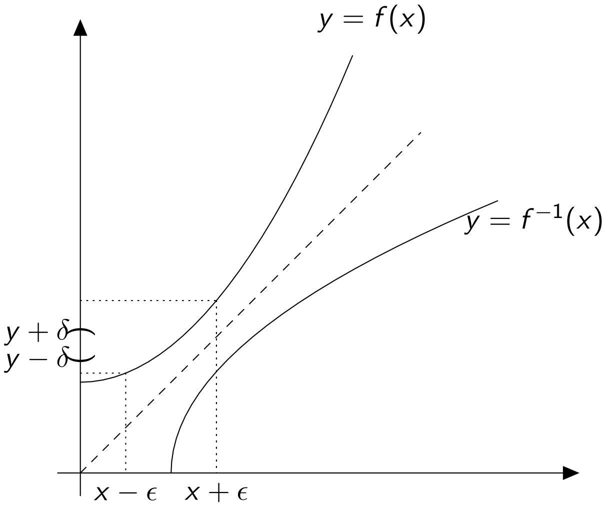


Figure: The continuity of the inverse function. For a given ϵ, we can take δ.

Roots and power function

Let us consider $f(x) = x^n$ defined on $\mathbb{R}_+ \cup \{0\}$. This is monotonically increasing (because, if $x_1 < x_2$, then $x_2^n = (x_1 + (x_2 - x_1))^n > x^n$ by the binomial theorem). Therefore, we can define the inverse function $f^{-1}(x)$ and denote it by $x^{\frac{1}{n}}$. This shows that, for any $x \in \mathbb{R}_+ \cup \{0\}$, there is one and only one y such that $y^n = x$. The function $g(x) = x^{\frac{1}{n}}$ is monotonically increasing and continuous by the previous Theorem.

Roots and power function

Let $p, q \in \mathbb{N}$, $x \geq 0$. Note that we have $(x^p)^q = x^{pq} = (x^q)^p$. Then it is easy to see that $(x^p)^{\frac{1}{q}} = (x^{\frac{1}{q}})^p$: if $y = (x^p)^{\frac{1}{q}}$, then $y^q = x^p = ((x^{\frac{1}{q}})^q)^p = (x^{\frac{1}{q}})^{pq}$ and hence $y = (x^{\frac{1}{q}})^p$. Furthermore, let $m \in \mathbb{N}$. Then for $y = (x^{mp})^{\frac{1}{mq}}$ we have $(y^q)^m = y^{mq} = x^{mp} = (x^1)^m$, and hence $y^q = x^p$ and $y = (x^p)^{\frac{1}{q}}$.

Therefore, we can write $y = x^{\frac{p}{q}}$ and no confusion arises.

Exercises

- Let $f(x) = x^4 + 3x^3 - x - 2$. Show that the equation $f(x) = 0$ has at least two solutions.
- Compute $\lim_{x \to 1} \sqrt{x + 3\sqrt{x}}$.
- Compute $\lim_{x \to 0} \frac{1 - \sqrt{1 - x^2}}{x^2}$.
- Compute $\lim_{x \to 0} \frac{\sqrt{1-x} - \sqrt{1+x}}{x}$.
- Consider $f(x) = x^2$. For $\epsilon = 0.1$, find a δ which shows the continuity of f at $x = 1$.
- Consider $f(x) = x^{\frac{1}{3}}$. For $\epsilon = 0.1$, find a δ which shows the continuity of f at $x = 0$.