Mathematical Analysis I: Lecture 10

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Start recording...

Announcements

- Today: Apostol Vol 1, Chapter 3.1-4.

Decimal representation of real numbers

Now that we have defined convergence of sequences, we can make sense of all decimal representations as real numbers.

Theorem

Let $a_n \in \mathbb{N}_0$ and $0 \leq a_n \leq 9$. Then $b_n = \sum_{k=0}^{n} a_k 10^{-k}$ converges to a real number.

Proof.

Let $b_n = \sum_{k=0}^{n} a_k 10^{-k}$. This is nondecreasing and bounded above by $a_0 + 1$. By Lemma of Monday, this converges to a real number. □

When the sequence converges, it converges to only one number. In this way, we can say that a decimal representation $a_0.a_1a_2a_3\cdots$ defines a real number.

Decimal representation of real numbers

Now we can prove that any repeating decimal representation gives a rational number. For example consider $0.123123123\cdots$. This can be written as

$$0.1 + 0.02 + 0.003 + 0.0001 + 0.00002 + 0.000003 + \cdots = \sum_{k=0}^{n} a_k 10^{-k},$$

where $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 1, a_5 = 2, a_6 = 3, \cdots$. It is easy to see that this is equal to
$0.123 + 0.000123 + \cdots = \sum_{k=1}^{n}(100a_{3k+1} + 10a_{3k+2} + a_{3k+3})1000^{-k}$. We know that this sum converges and compute

$$\sum_{k=1}^{n}(100a_{3k+1} + 10a_{3k+2} + a_{3k+3})1000^{-k} = 123\sum_{k=1}^{n} 1000^{-k}$$

$$\to 123\frac{1000^{-1}}{1 - 1000^{-1}} = \frac{123}{999}.$$

Decimal representation of real numbers

Theorem

Any real number given by a repeating decimal representation is rational.

Proof.

Indeed, let us take a repeating sequence $0 \leq a_n \leq 9$ as above. That is, there is $m \in \mathbb{N}$ such that $a_{n+m} = a_m$. Then, for $j, \ell \in \mathbb{N}$,

$$\begin{aligned}
\sum_{k=0}^{j\ell} a_k &= a_0 + \sum_{j=1}^{\ell} 10^{-jm} \sum_{k=1}^{m} a_k 10^{m-k} \\
&= a_0 + (\sum_{k=1}^{m} a_k 10^{m-k}) \frac{10^{-m}(1 - 10^{-j\ell})}{1 - 10^{-m}} \\
&\to a_0 + (\sum_{k=1}^{m} a_k 10^{m-k}) \frac{10^{-m}}{1 - 10^{-m}} = a_0 + (\sum_{k=1}^{m} a_k 10^{m-k}) \frac{1}{10^m - 1}
\end{aligned}$$

as $\ell \to \infty$. The last expression is evidently rational. □

Continuity of functions

Let us go back to studying functions. Among functions, we saw the sign function

$$\operatorname{sign} x := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

and its graph has a "jump" at $x = 0$.
Intuitively, the "jump" means that, the value at $x = 0$ is 0, but if one approaches to 0 from the right, the value of the function remains 1, while it is -1 from the left.

Continuity of functions

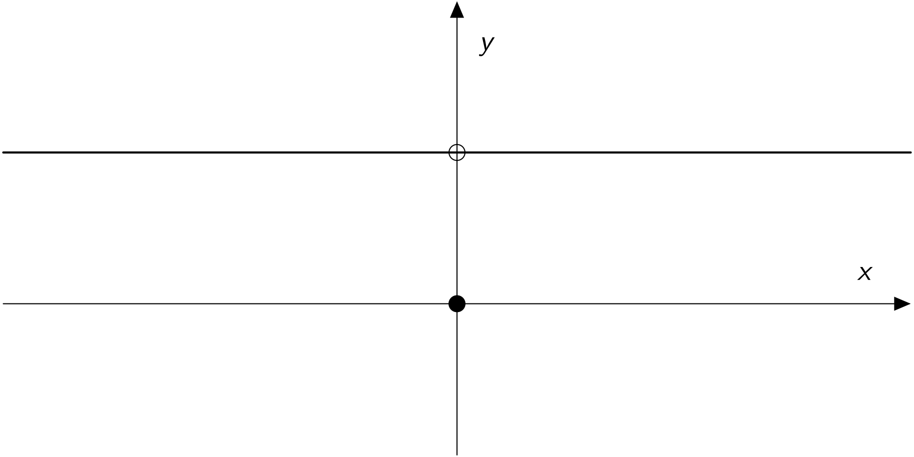

Figure: The graph of $y = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

Continuity of functions

Let us make this precise.

Definition

Let f be a function defined on S (the domain), and let $a \in \mathbb{R}$ such that there is a sequence $x_n \in S, x_n \neq a$ such that $x_n \to a$. We write

$$\lim_{x \to a} f(x) = L$$

if for any $\epsilon > 0$ there is $\delta > 0$ such that $|f(x) - L| < \epsilon$ for any $x \neq a, |x - a| < \delta$.

Continuity of functions

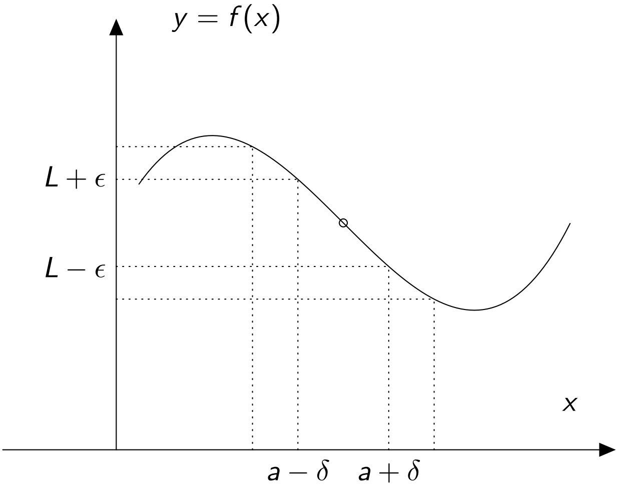

Figure: The limit $\lim_{x \to a} f(x)$.

Example

Let $f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

- Consider $a = 2$. Then, for any ϵ, we can take $\delta = \frac{1}{2}$ and $|f(x) - 1| = |1 - 1| = 0$ for any $x \in (2 - \delta, 2 + \delta) = (\frac{3}{2}, \frac{5}{2})$. Therefore, $\lim_{x \to 2} f(x) = 1$. A similar situation holds for any $x \neq 0$.
- Consider $a = 0$. Then, for any $x \neq 0$, $f(x) = 1$, hence again we have $\lim_{x \to 0} f(x) = 1$, although $f(0) = 0$ by definition.
- For the function $\operatorname{sign} x$, there is no limit $\lim_{x \to 0} f(x)$ at $x = 0$.

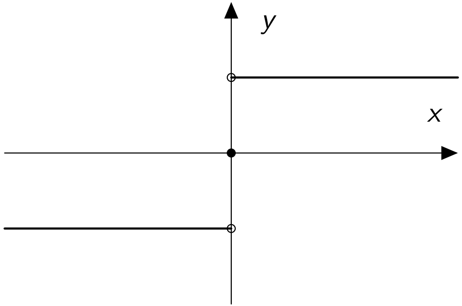

Figure: The graph of $y = \operatorname{sign} x$, with a "jump" at $x = 0$.

Continuity of functions

The limit makes precise the concept of "approaching a point". The absence of "jump" can also be formalized using limit.

Definition

Let f be a function defined on S (the domain), and let $a \in S$ (this time a is in the domain) such that there is a sequence $x_n \in S, x_n \neq a$ such that $x_n \to a$. We say that f is continuous at a if $\lim_{x \to a} f(x) = f(a)$. We say that f is continuous on S if it is continuous at each point in S.

Continuity of functions

Now we can understand the "jumps" in terms of limit and continuity.

Example

- The function $\operatorname{sign} x$ is not continuous at $x = 0$, because it does not have $\lim_{x\to 0} \operatorname{sign} x$.
- The function $f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is not continuous at $x = 0$, because $\lim_{x\to 0} f(x) = 1 \neq 0 = f(0)$.
- The function $f(x) = c$ is continuous. Indeed, let us fix $a \in \mathbb{R}$. For any ϵ, $|f(x) - c| = |c - c| = 0 < \epsilon$, hence $\lim_{x\to a} f(x) = c = f(a)$.
- The function $f(x) = x$ is continuous. Indeed, let us fix $a \in \mathbb{R}$. Then, for each $\epsilon > 0$, we take $\delta = \epsilon$ and for $|h| < \delta = \epsilon$ it holds that $|f(a + h) - a| = |a + h - a| = |h| < \delta = \epsilon$, therefore, $\lim_{x\to a} f(x) = a = f(a)$.

Continuity of functions

Theorem

Let f, g be functions defined on S, and let a such that there is $\{x_n\} \subset S$, $x_n \to a$. Assume that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then

- *There is $\delta > 0$ such that if $|x - a| < \delta, x \neq a$ then $|g(x)| \leq |M| + 1$.*
- $\lim_{x\to a}(f(x) + g(x)) = L + M$ *and* $\lim_{x\to a}(f(x)g(x)) = LM$.
- *If $M \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.*

Furthermore, if $a \in S$ and if f, g are continuous at a, then $f + g, fg$ are continuous at a. If $g(a) \neq 0$, then $\frac{f}{g}$ is continuous at a.

Continuity of functions

Proof.

The proof is similar to that of Theorem for sequences.

- Let $\delta > 0$ such that $|g(x) - M| < 1$ for x such that $|x - a| < \delta, x \neq a$. Then $|g(x)| < |M| + 1$.
- For a given $\epsilon > 0$, let $\delta > 0$ such that $|f(x) - L| < \frac{\epsilon}{2}, |g(x) - M| < \frac{\epsilon}{2}$ for $|x - a| < \delta, x \neq a$. Then $|f(x) + g(x) - L - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. For the product, for a given $\epsilon > 0$, let $\delta > 0$ such that $|f(x) - L| < \frac{\epsilon}{2(|M|+1)}, |g(x) - M| < \frac{\epsilon}{2(|L|+1)}$ and $|g(x)| < |M| + 1$ for $|x - a| < \delta, x \neq a$. Then $|f(x)g(x) - LM| =$ $|f(x) - L||g(x)| + |g(x) - M||L| < \frac{\epsilon(|M|+1)}{2(|M|+1)} + \frac{\epsilon|L|}{2(|L|+1)} < \epsilon$, which shows the desired limit.

Continuity of functions

Proof.

- We show that $\lim_{x\to a}\frac{1}{g(x)} = \frac{1}{M}$. Then the general case follows from this and the limit of product. Assume $M \neq 0$, and let $\epsilon > 0$. Then there is $\delta > 0$ such that $|g(x) - M| < \frac{|M|}{2}$ for $x \neq a, |x-a| < \delta$ and hence $|g(x)| > \frac{|M|}{2}$, in particular $g(x) \neq 0$. Now, there is $\tilde{\delta} > 0, \tilde{\delta} < \delta$ such that for $x \neq a, |x-a| < \tilde{\delta}$ it holds that $|g(x) - M| < \frac{\epsilon M^2}{2}$. Then

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|M - g(x)|}{|M||g(x)|} < \frac{\frac{\epsilon M^2}{2}}{\frac{M^2}{2}} = \epsilon,$$

  which shows the desired limit.

If f, g are continuous, then $\lim_{x\to a} f(x) = f(a), \lim_{x\to a} g(x) = g(a)$, hence $\lim_{x\to a}(f(x)+g(x)) = f(a)+g(a), \lim_{x\to a} f(x)g(x) = f(a)g(a), \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}$. □

Continuity of functions

From this, we know that

- If $f(x) = a_0 + a_1x^1 + \cdots a_nx^n$ (a polynomial), then f is continuous. $f(x) = x^2, f(x) = x^5 + 34x^3 - 454...$
- If $f(x) = \frac{P(x)}{Q(x)}$ and $P(x), Q(x)$ are polynomial, then f is continuous at x if $Q(x) \neq 0$. $f(x) = \frac{x-2}{x^2}$ is continuous on $x \neq 0$ (actually defined on $\{x \in \mathbb{R} : x \neq 0\}$, $f(x) = \frac{x^3}{x^2-1} = \frac{x^3}{(x-1)(x+1)}$ is continuous on $x \neq -1, 1$.

Exercises

- Let $x = 0.12341234\cdots$. Represent x as a rational number.
- Compute $\lim_{x\to 2} x^2$.
- Compute $\lim_{x\to 1} \frac{x+2}{x-3}$.
- Compute $\lim_{x\to -1} \frac{x^2+3x+2}{x^2-1}$.
- Let $f(x) = \begin{cases} x^2 & \text{if } x \geq 1 \\ 0 & \text{if } x \leq 0 \end{cases}$. Is f continuous or not? If not, where is it not continuous?
- Let $f(x) = \frac{x^2+3x+2}{x^2-1}$. Is f continuous or not? If not, where is it not continuous?