Mathematical Analysis I: Lecture 9

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05/10/2020

Start recording...

Announcements

- A supplemental course on mathematics (Trigonometry and Cartesian Geometry,Equalities and inequalities, Exponentials and logarithms, Radicals and absolute values)
- MS teams code: avc0vdz
- Starting on Tuesday 6 October

- Today: Apostol Vol 1, Chapter 10.2-3.

Convergence of sequences

We saw sequences of real numbers $a_1, a_2, \cdots$. A sequence can be infinite, that is, it continues infinitely. For example,

- $a_1 = 1, a_2 = 2$ and in general, $a_n = n$.
- $a_1 = 1, a_2 = 4$ and in general, $a_n = n^2$.

A sequence can be considered as a function with the domain $\mathbb{N}$.
Among sequences, we have seen the following:

- $a_1 = 1, a_2 = \frac{1}{2}$ and $a_n = \frac{1}{n}$.
- $a_1 = \frac{1}{2}, a_2 = \frac{3}{4}$ and $a_n = 1 - \frac{1}{2^n}$.

Intuitively, the first of them gets closer and closer to 0, while the second one gets closer and closer to 1. But what does it mean that it gets closer to a number?

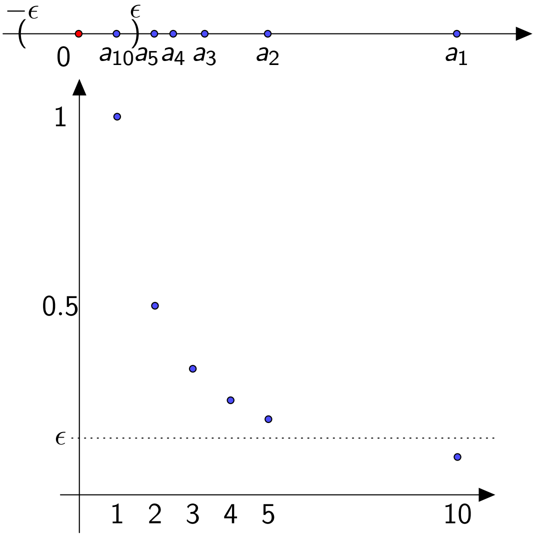

Figure: Up: the sequence $a_n = \frac{1}{n}$ plotted on the line. Bottom: the sequene $a_n = \frac{1}{n}$ as a function on $\mathbb{N}$.

Convergence of sequences

We make precise the notion that a sequence get "arbitrarily" close to a number as follows.

Definition

Let $\{a_n\}$ be a sequence of real numbers. If there is $L \in \mathbb{R}$ such that for each $\epsilon > 0$ there is N_ϵ such that for $n \geq N_\epsilon$ it holds that $|a_n - L| < \epsilon$, we say that $\{a_n\}$ **converges** to L.

We write this situation as $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$.

Convergence of sequences

Example

Let us see some convergent sequences.

- $a_1 = 1, a_2 = \frac{1}{2}$ and $a_n = \frac{1}{n}$. We expect that this sequence converges to 0. Indeed, for any $\epsilon > 0$, there is N_ϵ such that $\frac{1}{N_\epsilon} < \epsilon$ (the Archimedean property). Furthermore, if $n > N_\epsilon$, then $|\frac{1}{n} - 0| = \frac{1}{n} < \frac{1}{N_\epsilon} < \epsilon$, therefore, with $L = 0$, we have that $\{a_n\}$ converges to 0.
- $a_1 = \frac{1}{2}, a_2 = \frac{3}{4}$ and $a_n = 1 - \frac{1}{2^n}$. We expect that this sequence converges to 1. Indeed, for any $\epsilon > 0$, there is N_ϵ such that $\frac{1}{N_\epsilon} < \epsilon$ and note that $\frac{1}{2^{N_\epsilon}} < \frac{1}{N_\epsilon}$. Furthermore, if $n > N_\epsilon$, then $\frac{1}{2^n} < \frac{1}{N_\epsilon}$ and hence $|1 - \frac{1}{2^n} - 1| = \frac{1}{2^n} < \frac{1}{N_\epsilon} < \epsilon$, therefore, with $L = 1$, we have that $\{a_n\}$ converges to 1.
- The sequence $a_n = \frac{1}{\sqrt{n}}$ converges to 0. Indeed, for each ϵ, there is N_ϵ such that $\frac{1}{N_\epsilon} < \epsilon$, and hence if $n > N_\epsilon^2$, then $\frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N_\epsilon^2}} = \frac{1}{N_\epsilon} < \epsilon$.

Convergence of sequences

Note that

- If $\{a_n\}$ converges to L, then it does not converge to any other number. Indeed, if $x \neq L$, then take N such that $|a_n - L| < \frac{1}{2}|L - x|$ for $n > N$. Then by the triangle inequality $|L - x| < |a_n - x| + |a_n - L|$, and hence $|a_n - x| > |x - L| - |a_n - L| > \frac{1}{2}|L - x| \neq 0$. Therefore, $\{a_n\}$ does not converge to x.
- The sequence $a_1 = 1, a_2 = 0, a_3 = 1, \cdots, a_n = \frac{1}{2}(1 - (-1)^n)$ does not converge to any number.
- The sequence $a_1 = 1, a_2, \cdots, a_n = n$ does not converge to any number.
- In general, if for any x there is an $N_x \in \mathbb{N}$ such that for $n > N_x$ it holds that $|a_n| > x$, then we say that $\{a_n\}$ **diverges**.
- The sequence $a_n = 2^n$ diverges.

Convergence of sequences

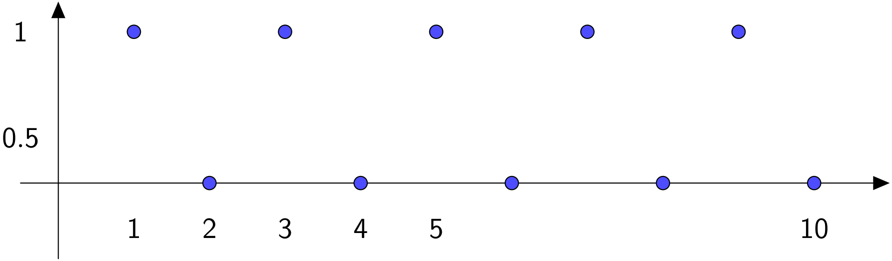

Figure: The sequence $a_n = \frac{1}{2}(1 - (-1)^n)$ as a function on $\mathbb{N}$.

Properties of convergent sequences

Given a sequence $\{a_n\}$, one can take a **subsequence** of it. That is, we take an increasing sequence of natural numbers $m_1 < m_2 < m_3 < \cdots$ and define a new sequence $b_n = a_{m_n}$.

Example

Given $a_n = \frac{1}{n}$ and $m_n = 2^n$, the subsequence is $a_{2^n} = \frac{1}{2^n}$.

If $\{a_n\}$ is convergent to L, then any subsequence $\{a_{m_n}\}$ is convergent to L. Indeed, as $m_1 < m_2 < m_3 \cdots$, we have $n \leq m_n$ and hence, for any $\epsilon > 0$, we take N such that $|a_n - L| < \epsilon$ for $n > N$, hence all $n > N$, $|a_{m_n} - L| < \epsilon$.

Properties of convergent sequences

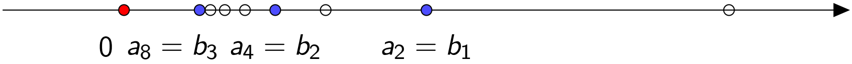

Figure: The subsequence a_{2^n} of the sequence $a_n = \frac{1}{n}$.

Properties of convergent sequences

We say that $\{a_n\}$ is **nondecreasing** (respectively **nonincreasing**) if $a_n \le a_{n+1}$ (respectively $a_n \ge a_{n+1}$) holds for all $n \in \mathbb{N}$. A sequence $\{a_n\}$ is said to be **bounded above** (respectively **bounded below**) if there is $M \in \mathbb{R}$ such that $a_n \le M$ (respectively $a_n \ge M$) for all $n \in \mathbb{N}$).

Lemma

Let $\{a_n\}$ be a nondecreasing sequence and bounded above. Then a_n converges to a certain real number $L \in \mathbb{R}$.

Proof.

Let $A = \{a_n : n \in \mathbb{N}\}$. As $\{a_n\}$ is bounded above, A is bounded above. We put $L = \sup A$. We know that, for each $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $L - \epsilon < a_N$. As a_n is nondecreasing, we have $L - \epsilon < a_n$ for all $n > N$. On the other hand, we have $a_n \le L$ because $L = \sup A$. Altogether, $|a_n - L| < \epsilon$ for such n. As n was arbitrary, a_n converges to L. $\square$

Properties of convergent sequences

Note that $|ab| = |a||b|$.

Lemma

The following hold.

- *If $a_n \to L$, then there is $\tilde{L}$ such that $|a_n| < \tilde{L}$ for all n.*
- *If $a_n \to L, b_n \to M$, then $a_n + b_n \to L + M, a_n \cdot b_n \to LM$. If $M \neq 0$, then $b_n \neq 0$ for sufficiently large n and $\frac{a_n}{b_n} \to \frac{L}{M}$.*
- *If $a_n > 0$ diverges, then $\frac{1}{a_n}$ converges to 0.*

Proof.

- Assume that $a_n \to L$. Given, say 1, there is N such that $|a_n - L| < 1$ for $n > N$, hence $|a_n| < L + 1$ for $n > N$. Then, we can take a number $\tilde{L}$ such that $|a_1|, \cdots, |a_{N-1}| < \tilde{L}$ and $L + 1 < \tilde{L}$.

Proof.

- Let $\epsilon > 0$ be arbitrary. There are $N_1, N_2 \in \mathbb{N}$ such that for $n > N_1$ (respectively $n > N_2$) it holds that $|a_n - L| < \frac{\epsilon}{2}$ (respectively $|b_n - M| < \frac{\epsilon}{2}$). Let N be the largest of N_1, N_2. Then we have

$$|a_n + b_n - L - M| \leq |a_n - L| + |b_n - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

  hence $a_n + b_n$ converges to $L + M$.
  As for the product, given $\epsilon > 0$, we take N such that $|a_n - L| < \frac{\epsilon}{2(|M|+1)}, |b_n - M| < \frac{\epsilon}{2|L|}$ and $|b_n| < |M| + 1$ for $n > N$ (this can be done as in the case of sum). Then

$$\begin{aligned} |a_n b_n - LM| &= |a_n b_n - b_n L + b_n L - LM| \\ &\leq |(a_n - L)b_n)| + |(b_n - M)L| \\ &\leq |a_n - L||b_n| + |b_n - M||L| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

  which shows the desired convergence.

Properties of convergent sequences

Proof.

- We prove $\frac{1}{b_n} \to M$. If $b_n \to M$ and $M \neq 0$, then $|b_n - M| < \frac{|M|}{2}$ for sufficiently large n, and hence $|b_n| > \frac{|M|}{2}$, in particular $b_n \neq 0$. We can now show that $\frac{1}{b_n} \to \frac{1}{M}$. Indeed, by taking N such that $|b_n - M| < \frac{\epsilon M^2}{2}$

$$\left|\frac{1}{b_n} - \frac{1}{M}\right| = \frac{|M - b_n|}{|M||b_n|} < \frac{\frac{\epsilon M^2}{2}}{\frac{M^2}{2}} = \epsilon,$$

  which shows $\frac{1}{b_n} \to M$. Now $\frac{a_n}{b_n} \to \frac{L}{M}$ follows from this and the product with a_n.
- For any $\epsilon > 0$, there is N such that for $n > N$ it holds that $|a_n| > \frac{1}{\epsilon}$, that is $\frac{1}{a_n} < \epsilon$, hence $\frac{1}{a_n}$ converges to 0.

$\square$

Powers

We denote $a^{-n} = \frac{1}{a^n}$.

Theorem

The following hold.

- *Let* $a > 1$. *Then* a^n *diverges.*
- *Let* $0 < a < 1$. *Then* a^n *converges to* 0.
- *Let* $0 < a < 1$. *Then* $b_n = \sum_{k=1}^{n} a^k$ *converges to* $\frac{a}{1-a}$.

Proof.

- If $a > 1$, we can write $a = 1 + y$ where $y > 0$. By the binomial theorem, we have

$$a^n = (1+y)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k y^{n-k} > 1 + ny,$$

  by only taking the terms $k = n, n-1$. Now it is clear that for any x there is large enough n such that $1 + ny > x$, therefore, $x < 1 + ny < a^n$, that is, a^n diverges.
- If $0 < a < 1$, then $\frac{1}{a} > 1$ and $(\frac{1}{a})^n$ diverges. Therefore, $a^n = (\frac{1}{a})^{-n}$ converges to 0.
- We know that $b_n = \sum_{k=1}^{n} a^k = \frac{a(1-a^n)}{1-a}$, and $a^n \to 0$.

$\square$

Exercises

- Let $a_n = \frac{1}{\sqrt{\sqrt{n}}}$ and $\epsilon = 0.01$. Find N such that for $n > N$ it holds $|a_n| < \epsilon$.
- Let $a_n = \frac{1}{2^n}$ and $\epsilon = 0.00001$. Find N such that for $n > N$ it holds $|a_n| < \epsilon$.
- Show that a constant sequence $a_n = C \in \mathbb{R}$ is convergent.
- Tell whether $\{a_n\}$ converges, and if it does, compute the limit $a_n = \frac{1}{1+\frac{1}{n}}$.
- Tell whether $\{a_n\}$ converges, and if it does, compute the limit $a_n = \frac{n}{1+n}$.
- Tell whether $\{a_n\}$ converges, and if it does, compute the limit $a_n = \frac{n^3+n^2+4}{n^3+100}$.