Mathematical Analysis I: Lecture 8

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02/10/2020

Start recording...

Announcements

- A supplemental course on mathematics (Trigonometry and Cartesian Geometry,Equalities and inequalities, Exponentials and logarithms, Radicals and absolute values)
- MS teams code: avc0vdz
- Starting on Tuesday 6 October

- If you do not understand the slides or notes, ask questions.
- If you find something strange, most probably you are right... I often make errors. Please point them out.

Exercises

Draw the set on the line $[-1, 2] \cup (-3, -2) \cup (0, 5]$.

Draw the set on the line $[-1,2] \cup (-3,-2) \cup (0,5]$.
Solution.

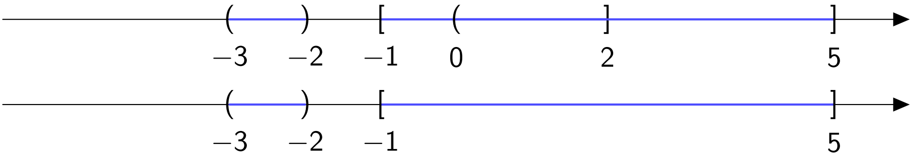

Figure: $[-1,2] \cup (-3,-2) \cup (0,5] = [-1,5] \cup (-3,-2)$.

Determine the inf and sup of $A = [-4, 0) \cup (2, 3)$.

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Solution. $\inf A = -4, \sup A = 3$.

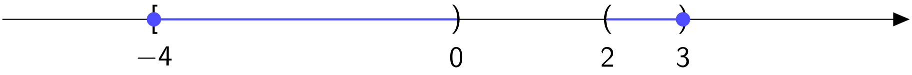

Figure: $[-4, 0) \cup (2, 3)$.

Determine the set $(1, 3) + (-2, 2]$.

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Solution. $(-1,5)$.

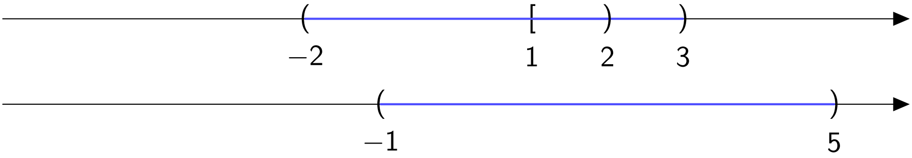

Figure: $(1,3)+(-2,2]=(-1,5)$.

Determine the set $5 \cdot (2, 3)$.

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Solution. $(10, 15)$.

Represent the set $\{x \in \mathbb{R} : x^2 - 2x < 0\}$ as an interval.

Represent the set $\{x \in \mathbb{R} : x^2 - 2x < 0\}$ as an interval.
Solution. The condition $x^2 - 2x < 0$ can be written equivalently as

$$\begin{aligned} x^2 - 2x < 0 &\iff x(x-2) < 0 \\ &\iff (x < 0, x - 2 > 0) \text{ or } (x > 0, x - 2 < 0) \\ &\iff (x < 0, x > 2) \text{ or } (x > 0, x < 2) \\ &\iff (0 < x < 2) \end{aligned}$$

hence it is the interval $(0, 2)$.

Represent the set $\{x \in \mathbb{R} : x^2 - 5x + 6 > 0\}$ as a union of intervals.

Represent the set $\{x \in \mathbb{R} : x^2 - 5x + 6 > 0\}$ as a union of intervals.
Solution. The condition $x^2 - 5x + 6 > 0$ can be written equivalently as

$$
\begin{aligned}
x^2 - 5x + 6 > 0 &\Longleftrightarrow (x-2)(x-3) > 0 \\
&\Longleftrightarrow (x - 2 < 0, x - 3 < 0) \text{ or } (x - 2 > 0, x - 3 < 0) \\
&\Longleftrightarrow (x < 2, x < 3) \text{ or } (x > 2, x > 3) \\
&\Longleftrightarrow x < 2 \text{ or } x > 3
\end{aligned}
$$

hence it is the union $(-\infty, 2) \cup (3, \infty)$.

Determine the decimal representation of $\frac{3}{7}$.

Determine the decimal representation of $\frac{3}{7}$.
Solution. Let $\frac{3}{7} = a_0.a_1a_2\cdots$. Note that $\frac{3}{7} < 1$, hence we have $a_0 = 0$.
Next, $\frac{3}{7} \times 10 = \frac{30}{7} = 4 + \frac{2}{7}$, hence we have $a_1 = 4$. Next,
$\frac{2}{7} \times 10 = \frac{20}{7} = 2 + \frac{6}{7}$, hence we have $a_2 = 2$, and so on.
Therefore, we have $\frac{3}{7} = 0.428571428571\cdots$.

		0.	4	2	8	5	7	1
7	)	3						
		0						
		3	0					
		2	8					
			2	0				
			1	4				
				6	0			
				5	6			
					4	0		
					3	5		
						5	0	
						4	9	
							1	0
								7
								3

Give an algorithm to produce a nonrepeating decimal representation.

Give an algorithm to produce a nonrepeating decimal representation.
Solution. Just an example. Set $0.101001000100001000001\cdots$.

Compute $\sum_{k=1}^{5}(2k+1)$.

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Solution.

$$\sum_{k=1}^{5}(2k+1) = (2+1)+(4+1)+(6+1)+(8+1)+(10+1)$$
$$= 3+5+7+9+11 = 35.$$

Compute $\sum_{k=2}^{6}(2(k-1)+1)$.

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Solution.

$$\sum_{k=1}^{5}(2(k-1)+1) = (2+1)+(4+1)+(6+1)+(8+1)+(10+1)$$
$$= 3+5+7+9+11 = 35.$$

Prove the formula $\sum_{k=1}^{n}(2k-1) = n^2$.

Prove the formula $\sum_{k=1}^{n}(2k-1) = n^2$.
Solution. By induction. For $n = 1$, we have $\sum_{k=1}^{1}(2k-1) = 1 = 1^2$.
Assuming the formula for n, we compute

$$\begin{aligned}\sum_{k=1}^{n+1}(2k-1) &= \sum_{k=1}^{n}(2k-1) + (2(n+1)-1) \\ &= n^2 + 2n + 1 = (n+1)^2.\end{aligned}$$

Write $\sum_{k=1}^{n}(2k-1)$ as a sum from $k=0$ to $n-1$.

Write $\sum_{k=1}^{n}(2k-1)$ as a sum from $k=0$ to $n-1$.
Solution. $\sum_{k=0}^{n-1}(2(k+1)-1) = \sum_{k=0}^{n-1}(2k+1)$.

Compute the sum $\sum_{k=1}^{n} 10^{-k}$.

Compute the sum $\sum_{k=1}^{n} 10^{-k}$.
Solution. By the formula for the sum of powers, we have

$$\frac{0.1(1-0.1^n)}{1-0.1} = \frac{0.0\underbrace{9\cdots9}_{n\text{-times}}}{0.9} = 0.\underbrace{1\cdots1}_{n\text{-times}}.$$

$$\sum_{k=1}^{n} 10^{-k} = 0.1 + 0.01 + 0.001 \cdots 0.\underbrace{0\cdots0}_{n-1\text{-times}}1 = 0.\underbrace{1\cdots1}_{n\text{-times}}.$$

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Solution. By the formula for the sum of powers, we have

$$\sum_{k=1}^{n} 2^{-1} = \frac{\frac{1}{2}(1-(\frac{1}{2})^n)}{1-\frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^n.$$

$$\sum_{k=1}^{1} 2^{-1} = \frac{1}{2}.$$

$$\sum_{k=1}^{2} 2^{-1} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

$$\sum_{k=1}^{3} 2^{-1} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}.$$

Expand $(x + y)^5$.

Expand $(x+y)^5$.

Solution. By the binomial theorem,

$$\begin{aligned}
&(x+y)^5 \\
&= \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5 \\
&= \frac{5!}{0!5!}x^5 + \frac{5!}{1!4!}x^4y + \frac{5!}{2!3!}x^3y^2 + \frac{5!}{3!2!}x^2y^3 + \frac{5!}{4!1!}xy^4 + \frac{5!}{5!0!}y^5 \\
&= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.
\end{aligned}$$

Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.
Solution. By the binomial theorem,
$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$.

Prove that $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.

Prove that $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.
Solution. By the binomial theorem,
$0 = 0^n = ((-1)+1)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k 1^{n-k} = \sum_{k=0}^{n}\binom{n}{k}(-1)^k$.

Determine the domains of the following $f(x) = \sqrt{x^2 - 1}$

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Solution. To have the square root, the number must be positive or zero. That is,

$$\begin{aligned} x^2 - 1 \geq 0 &\iff (x-1)(x+1) \geq 0 \\ &\iff (x-1 \geq 0, x+1 \geq 0) \text{ or } (x-1 \leq 0, x+1 \leq 0) \\ &\iff (x \geq 1, x \geq -1) \text{ or } (x \leq 1, x \leq -1) \\ &\iff (x \geq 1) \text{ or } (x \leq -1) \end{aligned}$$

hence the domain is $(-\infty, -1] \cup [1, \infty)$.

Determine the domains of the following $f(x) = \frac{1}{x^3+2x^2-x-2}$

Determine the domains of the following $f(x) = \frac{1}{x^3+2x^2-x-2}$
Solution. To have the division, the denominator must not be zero. That is,

$$x^3 + 2x^2 - x - 2 \neq 0 \iff (x+2)(x^2-1) \neq 0$$
$$\iff (x+2)(x-1)(x+1) \neq 0$$

hence the domain is $(-\infty, -2) \cup (-2, -1) \cup (-1, 1) \cup (1, \infty)$.

Determine the inverse functions of the following. $f(x) = x + 1$

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Solution. The inverse f^{-1} should satisfy $f^{-1}(x+1) = x$. We see that $f^{-1}(x) = x - 1$. Then we indeed have $(x-1)+1 = x$.

Determine the inverse functions of the following. $f(x) = \frac{1}{x}$ on $(0, \infty)$.

Determine the inverse functions of the following. $f(x) = \frac{1}{x}$ on $(0, \infty)$.
Solution. The inverse f^{-1} should satisfy $f^{-1}(\frac{1}{x}) = x$. We see that $f^{-1}(x) = \frac{1}{x}$. Then we indeed have $\dfrac{1}{\frac{1}{x}} = x$.

Compare the graphs. How can one obtain one from the other?
$f(x) = x^2, g(x) = (x - 1)^2 + 2.$

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$f(x) = x^2, g(x) = (x-1)^2 + 2$.
Solution. g can be obtained by shifting f by $(1,2)$. Indeed, their graphs are

$$\begin{aligned} f &= \{(x,y) \in \mathbb{R} \times \mathbb{R} : y = x^2\} \\ g &= \{(x',y') \in \mathbb{R} \times \mathbb{R} : y' = (x'-1)^2 + 2\} \\ &= \{(x',y') \in \mathbb{R} \times \mathbb{R} : y' - 2 = (x'-1)^2\} \end{aligned}$$

Therefore, if the point (x,y) is on the graph of f, then the point $(x',y') = (x+1, y+2)$ is on the graph of g.

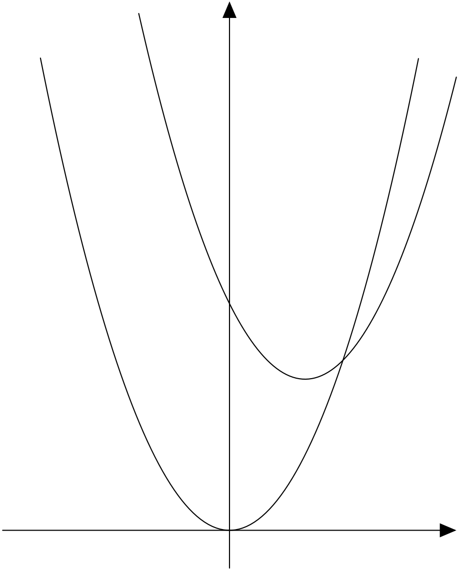

Figure: The graphs of $y = x^2$ and $y = (x-1)^2 + 2$.

Compare the graphs. How can one obtain one from the other?
$f(x) = \frac{1}{2}x^3 - x, g(x) = \frac{x^3}{16} - \frac{x}{2}$.

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$f(x) = \frac{1}{2}x^3 - x, g(x) = \frac{x^3}{16} - \frac{x}{2}$.
Solution. $g(x) = \frac{1}{2}(\frac{x}{2})^3 - \frac{x}{2}$ can be obtained by dilating the x-direction of f by 2. Indeed, their graphs are

$$f = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \frac{1}{2}x^3 - x\}$$
$$g = \{(x', y') \in \mathbb{R} \times \mathbb{R} : y' = \tfrac{1}{2}(\tfrac{x'}{2})^3 - \tfrac{x'}{2}\}$$

Therefore, if the point (x, y) is on the graph of f, then the point $(x', y') = (2x, y)$ is on the graph of g.

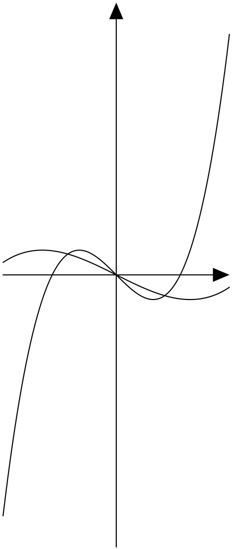

Figure: The graphs of $y = \frac{1}{2}x^3 - x$ and $y = \frac{x^3}{16} - \frac{x}{2}$.

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$f(x) = x^3 - x, g(x) = \frac{x^3 - x}{2}$
Solution. $g(x)$ can be obtained by dilating the y-direction of f by $\frac{1}{2}$.
Indeed, their graphs are

$$\begin{aligned} f &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^3 - x\} \\ g &= \{(x', y') \in \mathbb{R} \times \mathbb{R} : y' = \tfrac{1}{2}(x'^3 - x')\} \\ &= \{(x', y') \in \mathbb{R} \times \mathbb{R} : 2y' = x'^3 - x'\} \end{aligned}$$

Therefore, if the point (x, y) is on the graph of f, then the point $(x', y') = (x, \frac{y}{2})$ is on the graph of g.

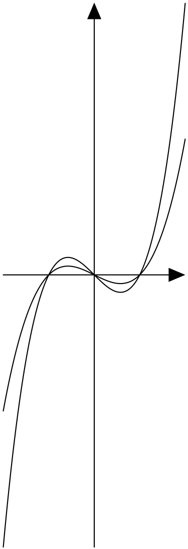

Figure: The graphs of $y = x^3 - x$ and $y = \frac{1}{2}(x^3 - x)$.

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$f(x) = \sqrt{1-x^2}, g(x) = \frac{1}{3}\sqrt{1-4(x+2)^2}$
Solution. $g(x)$ can be obtained by dilating the y-direction of f by $\frac{1}{3}$ and by dilating by $\frac{1}{2}$ then shifting the x-direction by -2 . Indeed, their graphs are

$$\begin{aligned} f &= \{(x,y) \in \mathbb{R}\times\mathbb{R} : y = \sqrt{1-x^2}\} \\ g &= \{(x',y') \in \mathbb{R}\times\mathbb{R} : y' = \tfrac{1}{3}\sqrt{1-4(x'+2)^2}\} \\ &= \{(x',y') \in \mathbb{R}\times\mathbb{R} : 3y' = \sqrt{1-(2(x'+2))^2}\} \end{aligned}$$

Therefore, if the point (x,y) is on the graph of f, then the point $(x',y') = (\frac{x}{2}-2, \frac{y}{3})$ is on the graph of g.

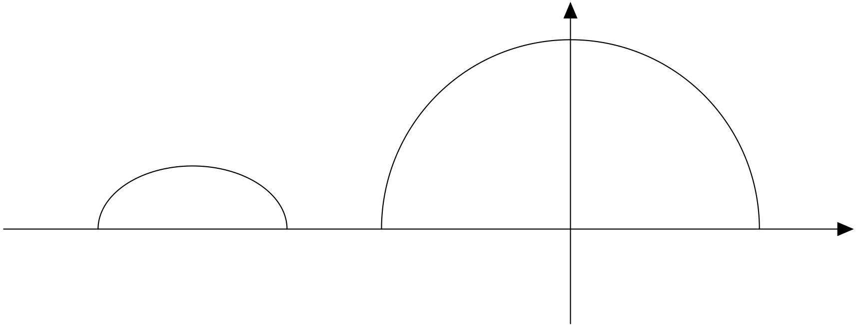

Figure: The graphs of $y = \sqrt{1 - x^2}$ and $y = \frac{1}{3}\sqrt{1 - 4(x+2)^2}$.