Mathematical Analysis I: Lecture 7

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01/10/2020

Start recording...

Tips

- Today: Apostol Vol. 1, Chapters 1.1-4

Functions

By a function we usually mean "a map" which assigns to a number x another number $f(x)$, or an assignment

$$x \longmapsto f(x).$$

There are many "real-world" examples of functions: When a quantity changes with time, you can use x as time (or often you denote it by t) and the quantity by $f(x)$. Or we can plot a set of data that depend on a parameter (more concretely: you take a path on a mountain and set x as the horizontal distance from the house and $f(x)$ as the height at the point x).

Functions

More precisely, we can consider it as follows: for each number x there is another number $f(x)$, and nothing else. We can express this situation using ordered pairs.

Let us assume that we know the correspondence $x \longmapsto f(x)$, defined on a subset ("domain") S. Then we can draw the graph, namely, the subset $\{(x, y) \in S \times \mathbb{R} : y = f(x)\}$, or in other words, we collect all points (x, y) where $y = f(x)$.

More generally we can define a **function** to be a subset f of $\mathbb{R} \times \mathbb{R}$ such that for each $x \in f$ there is one and only one y. Also in this case we denote the relation by $y = f(x)$. In this sense, the graph and the function are the same thing.

Functions

Let us introduce some terminology.

- $\{x \in \mathbb{R} :$ there is some $(x, y) \in f\}$ is called the **domain** of f.
- $\{y \in \mathbb{R} :$ there is some $(x, y) \in f\}$ is called the **range** of f.

Example

- $f(x) = x$. Namely, $f = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x\}$. The domain is $\mathbb{R}$, the range is $\mathbb{R}$.
- $f(x) = x^2$. Namely, $f = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2\}$. The domain is $\mathbb{R}$, the range is $[0, \infty)$.

Functions

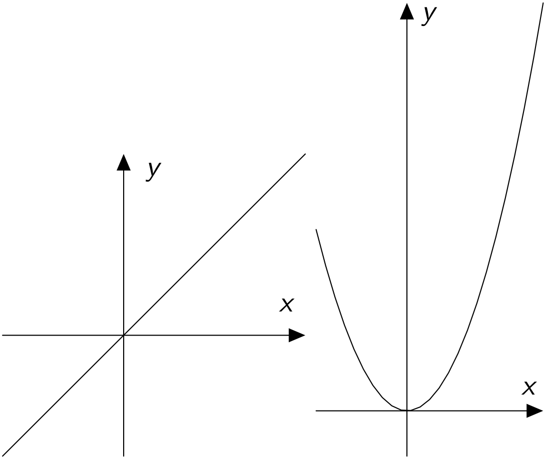

Figure: Left: the graph of $y = x$. Right: the graph of $y = x^2$.

Functions

Example

- $f(x) = x^5 - 2x^3 + 1$. $f = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^5 - 2x^3 + 1\}$. The domain is $\mathbb{R}$, the range is $\mathbb{R}$.
- $f(x) = \sqrt{x}$ for $x \geq 0$. Namely, $f = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq 0, y = \sqrt{x}\}$. The domain is $[0, \infty)$, the range is $[0, \infty)$.

Functions

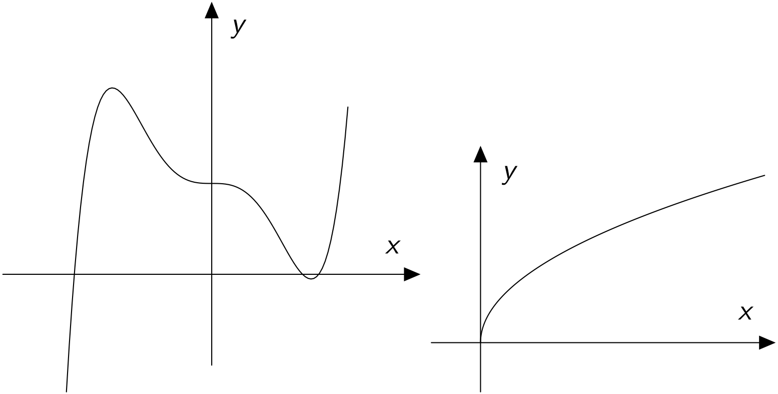

Figure: Left: the graph of $y = x^5 - 2x^3 + 1$. Right: the graph of $y = \sqrt{x}$.

Functions

Example

- $f(x) = \sqrt{1-x}$ for $1 - x \geq 0$, or $x \leq 1$. Namely, $f = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \leq 1, y = \sqrt{x-1}\}$. The domain is $(-\infty, 1]$, the range is $[0, \infty)$.
- The set $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$ is *not* a function. Indeed, for each $x \in (-1, 1)$, there are two numbers $y = \sqrt{1-x^2}, -\sqrt{1-x^2}$ that satisfy the equation $x^2 + y^2 = 1$.

Functions

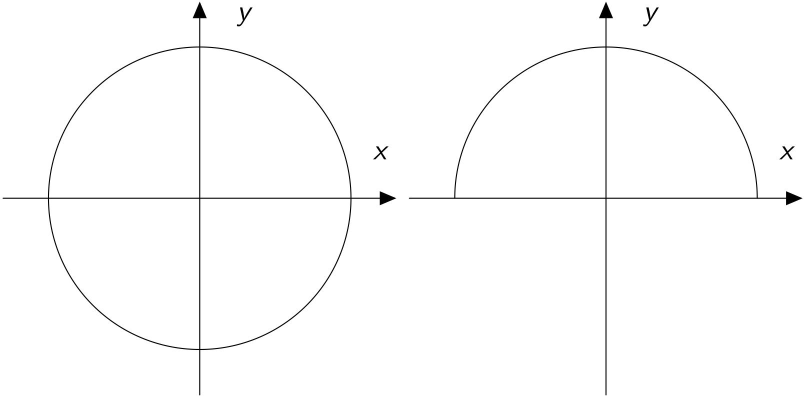

Figure: Left: the graph of $x^2 + y^2 = 1$, not a function of x. Right: the graph of $y = \sqrt{1 - x^2}$.

Functions

Example

- Let us introduce the **absolute value** of $x \in \mathbb{R}$:

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

  where $:=$ means we define the left-hand side by the right-hand side. This is also a function with the domain $\mathbb{R}$ and the range $[0, \infty)$.

- We define the **sign** of $x \in \mathbb{R}$:

$$\operatorname{sign} x := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

- We define the factorial of $x \in \mathbb{N}_0$: $f(n) = n!$. The domain is $\mathbb{N}_0$.

Functions

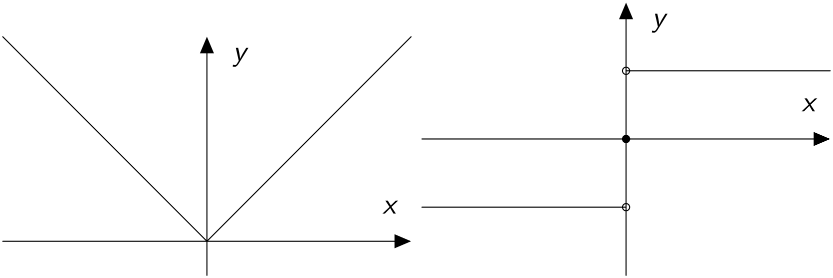

Figure: Left: the graph of $y = |x|$. Right: the graph of $y = \operatorname{sign} x$, with a “jump” at $x = 0$.

Operations on functions

When we have two or more functions, we can produce more functions. Let $f(x)$ be a function with domain S and $g(x)$ a function with domain T.

- Sum. We can define the sum $h(x) = f(x) + g(x)$, defined on $S \cap T$.
  Example: with $f(x) = x, g(x) = x^2, h(x) = x + x^2$.
- Product. We can define the product $h(x) = f(x) \cdot g(x)$, defined on $S \cap T$.
  Example: with $f(x) = x, g(x) = x^2, h(x) = x^3$.
- Division. We can define the division $h(x) = \frac{f(x)}{g(x)}$ defined on $S \cap \{x \in T : g(x) \neq 0\}$.
  Example: with
  $f(x) = x + 1, g(x) = (x + 2)(x - 1), h(x) = \frac{x+1}{(x+2)(x-1)}$, defined on $\mathbb{R} \setminus \{1, -2\} = (-\infty, -2) \cup (-2, 1) \cup (1, \infty)$.

Functions

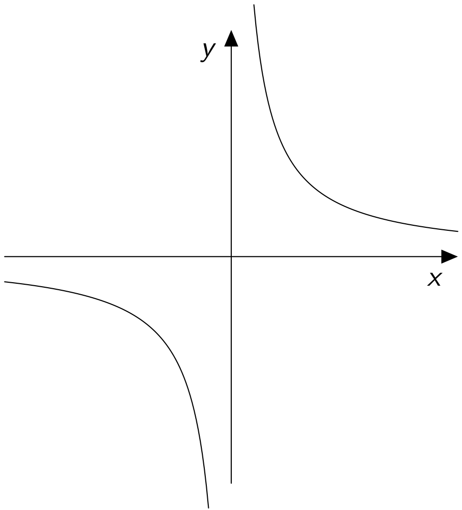

Figure: The graphs of $y = \frac{1}{x}$ on $(-\infty, 0) \cup (0, \infty)$.

Operations on functions

When we have two or more functions, we can produce more functions. Let $f(x)$ be a function with domain S and $g(x)$ a function with domain T.

- Composition. We can define the composed function $h(x) = f(g(x))$, defined on $\{x \in T : g(x) \in S\}$.
  Example: with $f(x) = \sqrt{x}, g(x) = x + 1, h_1(x) = \sqrt{x+1}$, defined on $\{x \in \mathbb{R} : x + 1 \geq 0\}$.
  Note that this is different from the composition in the reversed order: $h_2(x) = g(f(x)) = \sqrt{x} + 1$, defined on $[0, \infty)$.

Operations on functions

We say that a function $f(x)$ is **injective** if for any pair $x_1 \neq x_2$ in the domain, it holds that $f(x_1) \neq f(x_2)$. Similarly, we say that a function $f(x)$ is **surjective** if the range is $\mathbb{R}$. A function which is both injective and surjective is said to be **bijective**.

For example, $f(x) = x$ is injective and surjective (hence bijective), but $f(x) = x^2$ is neither injective nor surjective. But if we consider $f(x) = x^2$ with the restricted domain $[0, \infty)$, it is injective: for positive numbers $x_1 \neq x_2$, $x_1^2 \neq x_2^2$.

For an injective function $f(x)$, we can define the **inverse function** f^{-1}: the domain of f^{-1} is the range R of f, and it assigns to $f(x)$ the number x: it is characterized by $f^{-1}(f(x)) = x$. Its graph (its formal definition) is given by $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x \in R, x = f(y)\}$. The range of f^{-1} is the domain of f.

For example, consider $f(x) = x^2$ on the domain $[0, \infty)$. The range of f is $[0, \infty)$, hence the domain of f^{-1} is $[0, \infty)$. For any $x \in [0, \infty)$, we should have $f^{-1}(f(x)) = f^{-1}(x^2) = x$, therefore, $f^{-1}(x) = \sqrt{x}$.

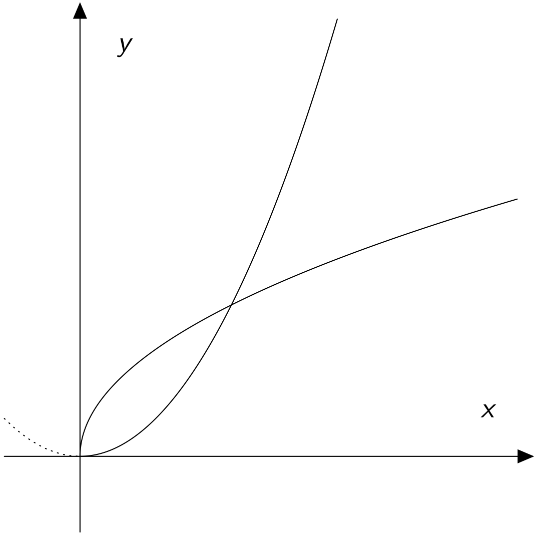

Figure: The graphs of $y = \sqrt{x}$ and $y = x^2$ on $[0, \infty)$.

The triangle inequality

Lemma

Let $x, a \in \mathbb{R}$, $a \geq 0$. Then $|x| \leq a$ if and only if $-a \leq x \leq a$.

Proof.

Assume that $x \geq 0$.

- If $|x| = x \leq a$, then $-a < 0 \leq x \leq a$.
- If $-a \leq x \leq a$, then $|x| = x \leq a$.

Instead, if we assume that $x < 0$, then

- If $|x| = -x \leq a$, then $-a \leq x < 0 \leq a$.
- If $-a \leq x \leq a$, then $|x| = -x \leq a$.

$\square$

The triangle inequality

Theorem

For any $x, y \in \mathbb{R}$, it holds that $|x+y| \leq |x| + |y|$.

Proof.

We have $-|x| \leq x \leq |x|, -|y| \leq y \leq |y|$ by Lemma, therefore, $-|x| - |y| \leq x + y \leq |x| + |y|$, and again by Lemma this implies that $|x+y| \leq |x| + |y|$. □

The triangle inequality

Corollary

For any $x_1, x_2, \cdots, x_n \in \mathbb{R}$, *it holds that* $|\sum_{k=1}^n a_k| \le \sum_{k=1}^n |a_k|$.

Proof.

By induction. For $n = 1$, $\left|\sum_{k=1}^1 a_1\right| = |a_1| = \sum_{k=1}^1 |a_k|$ is obvious. Assuming the inequality for n, we have

$$\begin{aligned}
\left|\sum_{k=1}^{n+1} a_k\right| = \left|\sum_{k=1}^{n} a_k + a_{n+1}\right| &\le \left|\sum_{k=1}^{n} a_k\right| + |a_{n+1}| && \text{by Theorem} \\
&\le \sum_{k=1}^{n} |a_k| + |a_{n+1}| = \sum_{k=1}^{n+1} |a_k| && \text{by induction hypothesis.}
\end{aligned}$$

which concludes the induction. □

Exercises

- Determine the domains of the following
  - $f(x) = \sqrt{x^2 - 1}$
  - $f(x) = \frac{1}{x^3+2x^2-x-2}$
- Determine the inverse functions of the following.
  - $f(x) = x + 1$
  - $f(x) = \frac{1}{x}$ on $(0, \infty)$.

  Compare the graphs. How can one obtain one from the other?
  - $f(x) = x^2, g(x) = (x-1)^2 + 2$.
  - $f(x) = x^3 - x, g(x) = \frac{x^3}{8} - \frac{x}{2}$.
  - $f(x) = x^3 - x, g(x) = \frac{x^3 - x}{2}$
  - $f(x) = \sqrt{1 - x^2}, g(x) = \frac{1}{3}\sqrt{1 - 4x^2}$