# Mathematical Analysis I: Lecture 6

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# Tips

- You solve exercises
- Today: Apostol Vol. 1, Chapter I.4

# Mathematical induction

The set $\mathbb{N}$ of natural numbers can be caracterized by the Peano axioms:

- $1 \in \mathbb{N}$
- For every $n \in \mathbb{N}$, $n + 1 \in \mathbb{N}$
- For every $n \in \mathbb{N}$, $n + 1 \neq 1$
- Let $S \subset \mathbb{N}$. If $1 \in S$ and $n + 1 \in S$ for any $n \in S$, then $S = \mathbb{N}$.

In other words, $\mathbb{N}$ consists of 1 and all other numbers obtained by adding 1 repeatedly to 1, and that is all. This is the precise definition of $\mathbb{N}$.

# Mathematical induction

With this characterization, we obtain the mathematical induction. Let $\varphi(n)$ be a set of propositions depending on $n \in \mathbb{N}$. If $\varphi(1)$ is true, and if we can prove $\varphi(n+1)$ from $\varphi(n)$, then $\varphi(n)$ is true for all natural numbers. Indeed, let $S = \{n \in \mathbb{N} : \varphi(n) \text{ is true }\}$. $S$ is a subset of $\mathbb{N}$, $1 \in S$ and if $n \in S$, then $n+1 \in S$. From the Peano axioms, we have $S = \mathbb{N}$. In other words, $\varphi(n)$ holds for all $n \in \mathbb{N}$.

# Mathematical induction

## Example

$n^2 \geq 2n - 1$ for all $n$.
Indeed, we apply mathematical induction to $\varphi(n) =$ "$n^2 \geq 2n - 1$". With $n = 1$, we have $1 \geq 2 \cdot 1 - 1 = 1$.
If we assume that this holds for $n$, then $(n+1)^2 = n^2 + 2n + 1 \geq 2n - 1 + 2n + 1 = 4n = 2n + 2n \geq 2n + 1 = 2(n+1) - 1$, therefore, we proved $\varphi(n+1)$ from $\varphi(n)$. We can now conclude that $\varphi(n)$ is true for all $n \in \mathbb{N}$.

# The well-ordering principle

We combine a proof by contradiction and mathematical induction.

**Theorem**

*For any nonempty subset $S \in \mathbb{N}$, there is the smallest element in $S$. That is, there is $n \in S$ such that $n \leq m$ for all $m \in S$.*

**Proof.**

Let us call **A** the assumption that $S$ is not empty.
Let us assume the contrary, that $S$ does not admit the smallest element (call this assumption **B**). It means that, for any $n \in S$, there is $m \in S$ such that $m < n$.
Let $T = \{n \in \mathbb{N} : m > n, \text{ for all } m \in S\}$. We show that $T = \mathbb{N}$ by induction.

## Proof.

- First, $1 \in T$. To prove this, assume that $1 \notin T$ (call this $\mathbf{C}_1$). Then, there must be $m \in S$ such that $m \leq 1$. This means $1 \in S$. But 1 is always the smallest element of any subset of $\mathbb{N}$, contradicting $\mathbf{B}$. Therefore, $\mathbf{C}_1$ is false and we obtain $1 \in T$.
- Next, let $n \in T$ and we prove that $n + 1 \in T$. Assume that $n + 1 \notin T$ (call this $\mathbf{C}_n$). Then, there is $m \in S$ such that $m \leq n + 1$, but since $n \in T$, it must hold that $n < m$. This means that $m = n + 1$, and any $\ell \leq n$ does not belong to $S$. Therefore, $m = n + 1$ would be the smallest element of $S$, contradicting $\mathbf{B}$. Therefore, $\mathbf{C}_n$ is false and we obtain $n + 1 \in T$.

Then by induction (the Peano axioms) we have $T = \mathbb{N}$. This implies that for any $m \in S$ it it holds for $m < n$ for all $n \in T = \mathbb{N}$. But there is no such $m$ (larger than any natural number), hence $S = \emptyset$. This contradicts the assumption **A** of the theorem. Therefore, the assumption **B** made in the proof is wrong. That is, $S$ admits the smallest element. □

# The well-ordering principle

## Corollary

*Let $x \in \mathbb{R}, x > 0$. There is $n \in \mathbb{N}$ such that $n - 1 \leq x < n$.*

## Proof.

By the Archimedean principle, there is $n$ such that $x < n$. Therefore, the set $\{m \in \mathbb{N} : x < m\}$ is nonempty, and by the well-ordering principle, it has the smallest element $n$. As this is the smallest element, $n - 1 \geq x$. □

We have used this property before to find the decimal representation of $x$.

# The summation and product notations

Assume that we have a sequence of numbers, that is a family $\{a_n\}_{n \in S}$ of real numbers indexed by $S \subset \mathbb{N}$. This means that we have numbers $a_1, a_2, a_3, \cdots$. Sometimes we start the index from 0, and have $a_0, a_1, a_2, \cdots$.

**Example**

- $a_1 = 1, a_2 = 2, a_3 = 3, \cdots$
- $a_1 = 1, a_2 = 4, a_3 = 9, \cdots$
- $a_1 = 4, a_2 = 2534, a_3 = \frac{3}{361}$ (a *finite* sequence stops at some $n \in \mathbb{N}$)

When we have a (finite) sequence, we can sum all these numbers up: $a_1 + \cdots + a_n$.

# The summation and product notations

When we have a (finite) sequence, we can sum all these numbers up: $a_1 + \cdots + a_n$.
We denote this by the following symbol.

$$\sum_{k=1}^{n} a_k = a_1 + \cdots + a_n$$

In this symbol, $k$ is a dummy index and plays no specific role. We have

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} a_{k+1} = a_1 + \cdots + a_n.$$

On the other hand, the number on the top ($n$ in this example) is where the sequence stops. Similarly, we can define $\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$ for $n \geq m$.

# The summation and product notations

More precisely, this is a recursive definition: We define $\sum_{k=1}^{1} = a_1$ and $\sum_{k=1}^{n+1} a_k = \sum_{k=1}^{n} a_k + a_{n+1}$ Similarly to mathematical induction, we define in this way $\sum_{k=1}^{n} a_k$ for all natural numbers $n \in \mathbb{N}$.

## Example

- $a_1 = 1, a_2 = 2, a_3 = 3$. $\sum_{k=1}^{3} a_k = 1 + 2 + 3 = 6$.
- $a_1 = 1, a_2 = 4, a_3 = 9, a_4 = 16$. $\sum_{k=1}^{4} a_k = 1 + 4 + 9 + 16 = 30$.

# The summation and product notations

Let us also introduce a symbol for product.

$$\prod_{k=1}^{n} a_k = a_0 \cdot a_1 \cdot \cdots \cdot a_n$$

## Example

- $a_1 = 1, a_2 = 2, a_3 = 3$. $\prod_{k=1}^{3} a_k = 1 \cdot 2 \cdot 3 = 6$.
- $a_1 = 1, a_2 = 4, a_3 = 9, a_4 = 16$. $\prod_{k=1}^{4} a_k = 1 \cdot 4 \cdot 9 \cdot 16 = 576$.

In particular, we denote

- For $a \in \mathbb{R}$, $a^n = \prod_{k=1}^{n} a$. For example, $a^1 = a, a^2 = a \cdot a, a^3 = a \cdot a \cdot a$. By convention, for $a \neq 0$, we set $a^0 = 1$.
- $n! = \prod_{k=1}^{n} k = 1 \cdot 2 \cdot \cdots \cdot n$. For example, $2! = 2, 3! = 6, 4! = 24, \cdots$. By convention, we set $0! = 1$.
- For $n, k \in \mathbb{N}, n \geq k$, we define $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. For example, $\binom{4}{2} = \frac{4!}{2!2!} = 6$.

# The summation formulas

## Theorem

*We have the following.*

- $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

## Proof.

We prove them by induction. $\sum_{k=1}^{1} k = 1 = \frac{1 \cdot 2}{2} = 1$ is correct. Assume the formula $\sum_{k=1}^{n} k = \frac{k(k+1)}{2}$ for $n$, then

$$\begin{aligned}\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ = \left(\frac{n}{2} + 1\right)(n+1) &= \frac{(n+2)(n+1)}{2}.\end{aligned}$$

Then by induction the formula holds for all $n \in \mathbb{N}$. □

# The summation formulas

## Theorem

*We have the following.* $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

## Proof.

We prove them by induction.

$\sum_{k=1}^{1} k^2 = 1^2 = \frac{1\cdot 2\cdot 3}{6} = 1$ is correct. Assume the formula for $n$, then

$$\begin{aligned}\sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \left(\frac{n(2n+1)}{6} + (n+1)\right)(n+1) = \frac{(2n^2+n+6n+6)(n+1)}{6} \\ &= \frac{(2n+3)(n+2)(n+1)}{6} = \frac{(2(n+1)+1)((n+1)+1)(n+1)}{6}.\end{aligned}$$

Then by induction the formula holds for all $n \in \mathbb{N}$. $\square$

# The summation formulas

## Theorem

*We have the following.*

- *For* $a \neq 1$, $\sum_{k=1}^{n} a^k = \frac{a(1-a^n)}{1-a}$.

## Proof.

We prove them by induction. $\sum_{k=1}^{1} a^k = a = \frac{a(1-a)}{1-a}$ is correct. Assume the formula $\sum_{k=1}^{n} a^k = \frac{a(1-a^n)}{1-a}$ for $n$, then

$$\begin{aligned}\sum_{k=1}^{n+1} a^k &= \left(\sum_{k=1}^{n} a^k\right) + a^{n+1} = \frac{a(1-a^n)}{1-a} + a^{n+1} \\ &= \frac{a - a^{n+1} + a^{n+1} - a^{n+2}}{1-a} = \frac{a(1-a^{n+1})}{1-a}\end{aligned}$$

Then by induction the formula holds for all $n \in \mathbb{N}$. $\square$

# The binomial theorem

## Lemma

$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ *for* $n \geq k$.

## Proof.

We prove this by induction, but in a slightly different form: we prove that the formula is correct for $n = k$, and prove that it holds for $n+1$ assuming the formula for $n$. In this way, we prove the formula for $n \geq k$.
If $n = k$, we have $\binom{k+1}{k} = \frac{(k+1)!}{k!(k+1-k)!} = k+1 = \frac{k!}{(k-1)!} + 1 = \binom{k}{k-1} + \binom{k}{k}$.

## Proof.

Assuming the formula for $n$, we have

$$\begin{aligned}
\binom{n+2}{k} &= \frac{(n+2)!}{k!(n+2-k)!} \\
&= \frac{n+2}{n+2-k}\binom{n+1}{k} \\
&= \frac{n+2}{n+2-k}\left(\binom{n}{k-1}+\binom{n}{k}\right) \\
&= \frac{n+2}{n+2-k}\left(\frac{n!}{(k-1)!(n-k+1)!}+\frac{n!}{k!(n-k)!}\right) \\
&= \frac{(n+1)!}{(k-1)!(n+2-k)!}+\frac{n!}{(k-1)!(n+2-k)!} \\
&\quad + \frac{n+2}{n+2-k}\cdot\frac{n!}{k!(n-k)!}
\end{aligned}$$

## Proof.

Assuming the formula for $n$, we have

$$\begin{aligned}
\binom{n+2}{k} &= \frac{(n+1)!}{(k-1)!(n+2-k)!} + \frac{n!}{(k-1)!(n+2-k)!} \\
&\quad + \frac{n+2}{n+2-k} \cdot \frac{n!}{k!(n-k)!} \\
&= \frac{(n+1)!}{(k-1)!(n+2-k)!} \\
&\quad + \frac{n!}{k!(n+1-k)!} \cdot \frac{k+(n+2)(n+1-k)}{n+2-k} \\
&= \frac{(n+1)!}{(k-1)!(n+2-k)!} + \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k-1} + \binom{n+1}{k}
\end{aligned}$$

and this concludes the induction. $\square$

# The binomial theorem

**Theorem**

*For any* $a, b \in \mathbb{R}, n \in \mathbb{N}$, $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$, *where in this theorem we mean* $0^0 = 1$.

**Proof.**

By induction. For $n = 0$, this holds in the sense of $1 = 1$.

## Proof.

Assume that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ holds. Then,

$$
\begin{aligned}
(a+b)^{n+1} &= (a+b)^n \cdot (a+b) = (a+b)\sum_{k=0}^{n}\binom{n}{k} a^k b^{n-k} \\
&= \sum_{k=0}^{n}\binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^{n}\binom{n}{k} a^k b^{n-k+1} \\
&= \sum_{k=1}^{n+1}\binom{n}{k-1} a^k b^{n+1-k} + \sum_{k=0}^{n}\binom{n}{k} a^k b^{n-k+1} \\
&= \sum_{k=1}^{n}\left(\binom{n}{k-1} + \binom{n}{k}\right) a^k b^{n+1-k} + a^{n+1}b^0 + a^0 b^{n+1} \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k} a^k b^{n+1-k}.
\end{aligned}
$$

$\square$

# The binomial theorem

For example, we have

- $(x+y)^2 = x^2 + 2xy + y^2$
- $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
- $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

and so on.

# Exercises

- Compute $\sum_{k=1}^{5}(2k+1)$
- Compute $\sum_{k=2}^{6}(2(k-1)+1)$
- Prove the formula $\sum_{k=1}^{n}(2k-1) = k^2$.
- Compute the sum $\sum_{k=1}^{n} 10^{-1}$.
- Compute the sum $\sum_{k=1}^{n} 2^{-1}$.
- Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.
- Prove that $\sum_{k=0}^{n}(-1)^k \binom{n}{k} = 0$.