Mathematical Analysis I: Lecture 5

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Start recording...

Exercise session

- I pick one of the exercises I proposed during the lecture
- You (try to) solve it
- Justify your solution by mathematical logic and theorems
- If anyone gets the answer, we are happy
- If not, I present a solution
- You try to make variations of the exercises and try to solve them

Prove $a(-b) = -ab$ from the axiom of the real numbers.

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Solution. We use

- definition of negative: $-x$ is the unique real number such that $x + (-x) = 0$
- distributive law $(x + y)z = xz + yz$
- definitions of zero and negative

Indeed, we have

$$\begin{aligned} ab + a(-b) &= a(b + (-b)) && \text{(distributive law)} \\ &= a \cdot 0 && \text{(definition of negative)} \\ &= 0 && \text{(definition of zero)} \end{aligned}$$

and by the definition of negative, $a(-b) = -(ab)$.

Prove that $a^{-1}b^{-1} = (ab)^{-1}$.

Prove $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

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Solution. We use

- commutativity and associativity of product
- definition of fraction $\frac{x}{y} = xy^{-1}$
- $a^{-1}b^{-1} = (ab)^{-1}$

Indeed, we have

$$\begin{aligned}
\frac{a}{b} \cdot \frac{c}{d} &= ab^{-1}cd^{-1} && \text{(definition of fraction)} \\
&= acb^{-1}d^{-1} && \text{(commutativity and associativity)} \\
&= ac(bd)^{-1} && \text{(exercise)} \\
&= \frac{ac}{bd}
\end{aligned}$$

Prove $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

For real numbers a, b, c, prove that if $a < b$ and $b < c$, then $a < c$.

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Solution. We use

- if $x < y$, then $x + z < y + z$
- if $0 < x, 0 < y$, then $0 < x + y$
- associativity

Indeed, we have $0 < b - a$ by adding $-a$ to $a < b$. Similarly, $0 < c - b$. By taking the sum, we get $0 < (b - a) + (c - b) = c - a$. Adding a to both side, we get $a < c$.

Let $A = \{0, 1, 2, 3\}, B = \{x \in \mathbb{Z} : \text{ there is } y \in \mathbb{Z} \text{ such that } x = 2y\}$.
What are $A \cap B$ and $A \cup B$?

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What are $A \cap B$ and $A \cup B$?
Solution. B is the set of even numbers, hence $A \cap B = \{0, 2\}$. $A \cup B$ does not have a nice representation, but formally it is $A \cup B = \{x \in \mathbb{Z} : x = 1, 3 \text{ or there is } y \in \mathbb{Z} \text{ such that } x = 2y\} = \{0, 1, 2, 3, 4, 6, 8, \cdots\}$.

Let $A = \{x \in \mathbb{Z} : \text{ there is } y \in \mathbb{Z} \text{ such that } x = 3y\}, B = \{x \in \mathbb{Z} :$ there is $y \in \mathbb{Z}$ such that $x = 2y\}$. What is $A \cap B$?

Let $A_n = \{x \in \mathbb{Z} : \text{ there is } y \in \mathbb{Z} \text{ such that } x = ny\}$. What are $\bigcap_{n\in\mathbb{N}, n\geq 2} A_n$ and $\bigcup_{n\in\mathbb{N}, n\geq 2} A_n$?

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Solution. Let $x \in \mathbb{Z}$. If $x = 0$, then $x \in A_n$ for all n, hence $x \in \bigcap_{n\in\mathbb{N},n\geq 2} A_n$. On the other hand, if $x \neq 0, -1$, then $x \notin A_{x+1}$, hence $x \notin \bigcap_{n\in\mathbb{N},n\geq 2} A_n$. If $x = -1$, $x \notin A_2$, hence $x \notin \bigcap_{n\in\mathbb{N},n\geq 2} A_n$. Altogether, $\bigcap_{n\in\mathbb{N},n\geq 2} A_n = \{0\}$.

For $x \not 1, -1$, $x \in A_x$ (if x is potivive) or $x \in A_{-x}$ (if x is negative). On the other hand, $1, -1 \notin A_n$ for any $n \geq 2$. Altogether, $\bigcup_{n\in\mathbb{N},n\geq 2} A_n = \{x \in \mathbb{Z} : x \neq 1, -1\}$.

Let $A = \{1, 2, 3, 4, 5, 6\}$, and $B = \{(x, y) \in A \times A : y = 2x\}$. Give all elements of B and draw its graph.

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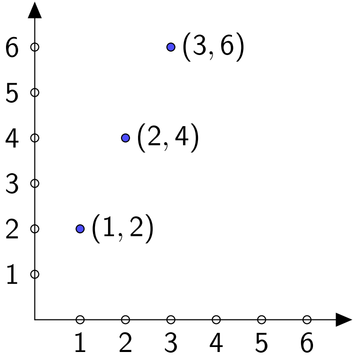
Solution. Check all $6 \times 6 = 36$ elements. See Figure, it is $\{(1, 2), (2, 4), (3, 6)\}$.

Notice that it is on a **straight line**!



Let $A = \mathbb{Z}$, and $B = \{(x, y) \in A \times A : y > x + 2\}$. Draw (a part of) its graph. What if $A = \mathbb{Q}, \mathbb{R}$?

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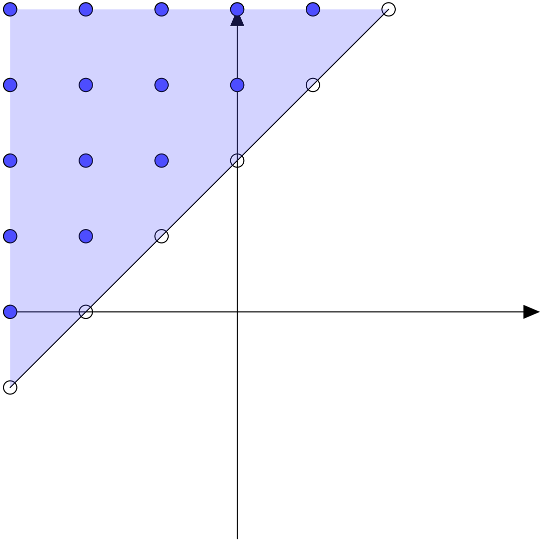
Solution.



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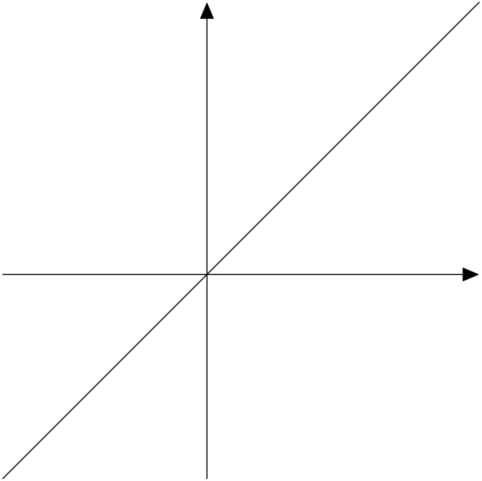

Figure: The set of all points $(x, y) \in \mathbb{R} \times \mathbb{R}$ with $y = x$.

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Solution. Note that

- if $x = 1$, $y = 1^2 = 1$.
- if $x = 0.5$, $y = 0.5^2 = 0.25$.
- if $x = 0.1$, $y = 0.1^2 = 0.01$.
- if $x = 2$, $y = 2^2 = 4$.
- if $x = 3$, $y = 3^2 = 9$.
- if $x = -1$, $y = (-1)^2 = 1$.

This is known as a **parabola**.

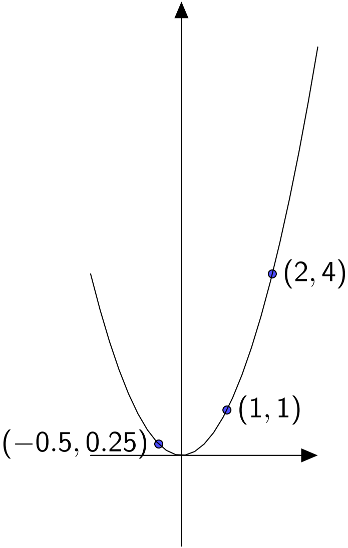

Figure: The set of all points $(x, y) \in \mathbb{R} \times \mathbb{R}$ with $y = x$.

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Solution. Note that one has to take the region below the parabola $y = x^2 + 1$.

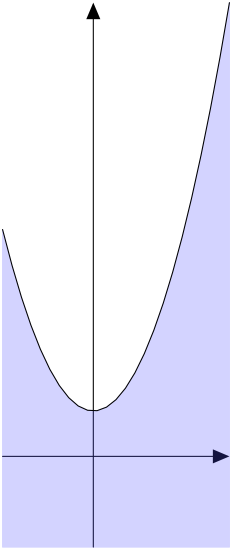

Figure: The set of all points $(x, y) \in \mathbb{R} \times \mathbb{R}$ with $y < x^2 + 1$.

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Solution 2. Use the fact that $\sqrt{2}$ is irrational. If $2\sqrt{2}$ were rational, then $2\sqrt{2} = \frac{p}{q}$ for some $p, q \in \mathbb{N}$, but this would imply that $\sqrt{2} = \frac{p}{2q}$ is rational, which is a contradiction. Therefore, $2\sqrt{2}$ must be irrational.

Prove that $\sqrt{3}$ is irrational.

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Solution. Follow the proof of irrationality of $\sqrt{2}$. Use the fact that x^2 is a multiple of 3 if and only if x is a multiple of 3 (why?).

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Solution. 1 is the largest element in A, hence $\sup A = 1$.
0 is a lower bound of A. On the other hand, for any $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ (the Archimedean principle). This means that any positive number ϵ cannot be a lower bound. Therefore, 0 is the greatest lower bound: $\inf A = 0$.

Let $A = \{0.9, 0.99, 0.999, \cdots\}$. Determine $\inf A$ and $\sup A$.

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Solution. 0.9 is the smallest element in A, hence $\inf A = 0.9$.
1 is an upper bound of A. On the other hand, for any $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ (the Archimedean principle). We can take $0.0\cdots01 < \frac{1}{n}$, and $1 - 0.0\cdots01 = 0.09\cdots99$, and the next element in A is larger than it. This means that for any positive number $1-\epsilon$ cannot be an upper bound Therefore, 1 is the least upper bound: $\sup A = 1$.

Let $B = \{0.3, 0.33, 0.333, \cdots\}$. Determine $\inf A$ and $\sup A$.

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Solution. We have $B = \frac{1}{3}A$, where A is the set in the previous exercise. It holds that $\sup B = \frac{1}{3}\sup A = \frac{1}{3}, \inf B = \frac{1}{3}\inf A = 0$ (why? See the proof of the theorem $\sup A + \sup B = \sup(A + B)$).

$x = 0.000001$. For which n does it hold that $\frac{1}{n} < x$?

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Solution. $x = 1/1000000$. So we can take $n = 1000001$.