Mathematical Analysis I: Lecture 4

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Start recording...

Announcement and corrections

- Office hour: Tuesday 10:00–11:00 (if this does not suit you, just drop me a line on Teams or via email)
- A make-up lecture: September 29th (exercises on rational and real numbers and sets)

- Take notes! Even though I upload slides and lecture notes, it is very useful for you to write formulas and graphs by yourselves. Try to follow the arguments in the proofs, compute the examples, do exercises.
- Today: Apostol Vol. 1, Chapter I.3.

- (commutativity) $x + y = y + x, x \cdot y = y \cdot x$

Corrections

Theorem

Let $A = \{x \in \mathbb{Q} : x^2 < 2\} (\subset \mathbb{R})$. *Then* A *is bounded above but* $\sup A \notin \mathbb{Q}$.

Proof.

A is bound above. Let $s = \sup A \in \mathbb{R}$. We prove $s^2 = 2$ by contradiction.

- if $s^2 < 2$, then we take $\epsilon > 0$ such that $0 < \epsilon < \frac{2-s^2}{s}$ (or $s\epsilon < 2 - s^2$) and $\epsilon < s$. Then $(s + \frac{\epsilon}{4})^2 = s^2 + s\frac{\epsilon}{2} + \frac{\epsilon^2}{16} < s^2 + \frac{s\epsilon}{2} + \frac{s\epsilon}{2} < s^2 + s\epsilon < 2$, therefore, s is not an upper bound of A (because $s + \frac{\epsilon}{4} \in A$), contradiction.
- if $s^2 > 2$, then we take $\epsilon > 0$ such that $0 < \epsilon < \frac{s^2-2}{s}$ (or $s\epsilon < s^2 - 2$) and $\epsilon < s$. Then $(s - \frac{\epsilon}{4})^2 = s^2 - s\frac{\epsilon}{2} + \frac{\epsilon^2}{16} > s^2 - s\epsilon > 2$, therefore, s is not the least upper bound of A (because $s - \frac{\epsilon}{4} \in A$ is another upper bound, smaller than s), contradiction.

But $s \in \mathbb{Q}$ never satisfies $s^2 = 2$. Hence $s = \sup A (= \sqrt{2}) \notin \mathbb{Q}$. □

Intervals

In the set of real numbers, we can consider **intervals**: let $a, b \in \mathbb{R}$ and $a < b$. We introduce

- $(a, b) = \{x \in \mathbb{R} : a < x, x < b\}$ (an **open** interval)
- $(a, b] = \{x \in \mathbb{R} : a < x, x \leq b\}$
- $[a, b) = \{x \in \mathbb{R} : a \leq x, x < b\}$
- $[a, b] = \{x \in \mathbb{R} : a \leq x, x \leq b\}$ (a **closed** interval)
- $(a, \infty) = \{x \in \mathbb{R} : a < x\}$
- $[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$
- $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$
- $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$

Remember that, a, b are given numbers, and x is a “dummy” number. You can write them in a different way, without using x:

- (a, b) is the set of all numbers larger than a and smaller than b
- $[a, b]$ is the set of all numbers larger than or equal to a and smaller than or equal to b

Intervals

Example

Consider $(0, 1)$.

- $0.1, 0.2, 0.5, 0.999 \in (0, 1)$.
- $0, 1, 2, 3, 10, -1, -2 \notin (0, 1)$.
- $\sup(0, 1) = 1$.
- $\inf(0, 1) = 0$.

Consider $[0, 1]$.

- $0, 0.1, 0.2, 0.5, 0.999, 1 \in (0, 1)$.
- $2, 3, 10, -1, -2 \notin (0, 1)$.
- $\sup(0, 1) = 1$.
- $\inf(0, 1) = 0$.

Intervals

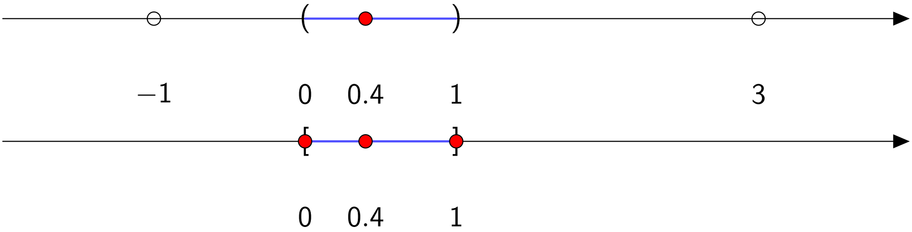

Figure: Open and closed intervals $(0, 1)$ and $[0, 1]$. The open interval does not include the edges $0, 1$, while the closed interval $[0, 1]$ does.

Operations on sets

Let A, B be subsets of $\mathbb{R}$ and $a \in \mathbb{R}$. We denote various subsets $\mathbb{R}$ as follows.

- $A + a = \{x \in \mathbb{R} : x = y + a \text{ for some } y \in A\} = \{y + a : y \in A\}$
- $A - a = \{x \in \mathbb{R} : x = y - a \text{ for some } y \in A\} = \{y - a : y \in A\}$
- $aA = \{x \in \mathbb{R} : x = ay \text{ for some } y \in A\} = \{ay : y \in A\}$
- $A + B = \{x \in \mathbb{R} : x = y + z \text{ for some } y \in A, z \in B\} = \{y + z : y \in A, z \in B\}$
- $A - B = \{x \in \mathbb{R} : x = y - z \text{ for some } y \in A, z \in B\} = \{y - z : y \in A, z \in B\}$
- $AB = \{x \in \mathbb{R} : x = yz \text{ for some } y \in A, z \in B\} = \{yz : y \in A, z \in B\}$

Operations on sets

We write $a < x < b$ as a shorthand notation for $a < x$ and $x < b$.

Example

- Consider $A = (0, 1)$, $x = 2$. Then $A + x = (2, 3)$, because if $0 < y < 1, 2 < y + 2 < 3$. Note that the boundary $2, 3$ is not included.
- Consider $A = [1, 2]$, $B = (2.4, 2.6)$. Then $A + B = (3.4, 4.6)$. Note that the boundary $2, 3$ is not included, because there is no $x \in A, y \in B$ such that $x + y = 3.4$ or 4.6.
- Consider $A = [-1, 1), a = 2$. Then $2A = [-2, 2)$.

Operations on sets

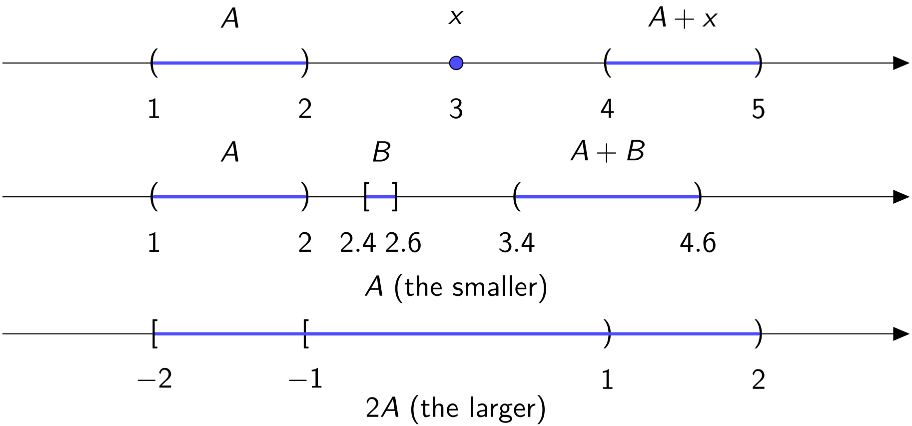

Figure: Intervals and their operations. Top: $(1, 2) + 3 = (4, 5)$. Middle: $(1, 2) + [2.4, 2, 6] = (3.4, 4.6)$. Bottom: $2[-1, 1) = [-2, 2)$.

Operations on sets

Let us have a set defined by specification: $A = \{x \in \mathbb{R} : \varphi(x)\}$, where $\varphi(x)$ is a statement on x. For example,
$A = \{x \in \mathbb{R} : x > 0, x < 1\} = (0, 1)$.
If you have another set $B = \{x \in \mathbb{R} : \psi(x)\}$ but $\psi(x)$ and $\varphi(x)$ are equivalent, then the two sets contain the same elements, hence are the same: $A = B$.

Example

- $0 < x < 1$ and $1 < x + 1 < 2$ are equivalent, therefore,
  $A = \{x \in \mathbb{R} : 0 < x < 1\} = \{x \in \mathbb{R} : 1 < x + 1 < 2\} = B(= (0, 1))$.
- $x^2 < 4$ and $-2 < x < 2$ are equivalent, therefore,
  $A = \{x \in \mathbb{R} : x^2 < 4\} = \{x \in \mathbb{R} : -2 < x < 2\} = B(= (-2, 2))$.

Exercise Represent the set $A = \{x \in \mathbb{R} : x^2 \geq 2\}$ as an interval.

Some properties of upper and lower bounds

Note that $\sup A, \inf A$ are only defined for nonempty sets (otherwise the definition is meaningless).

Lemma

If $x, y \in \mathbb{R}$ and $x - \epsilon < y$ for any $\epsilon > 0$, then $x \leq y$.

Proof.

By contradiction. If $x > y$, then by Archimedean property, we have n such that $\frac{1}{n} < x - y$, in other words, $x - \frac{1}{n} > y$, which contradicts the assumption that $x - \epsilon < y$ for arbitrary $\epsilon > 0$. □

Some properties of upper and lower bounds

Recall that, for $A \subset \mathbb{R}$ bounded above, we have shown that for any $\epsilon > 0$ there is $x \in A$ such that $\sup A - \epsilon < x$.

Theorem

Let $A, B \subset \mathbb{R}$ and define $C = A + B$.

- *if A, B are bounded above, then $A + B$ is bounded above and $\sup A + \sup B = \sup C$.*
- *if A, B are bounded below, then $A + B$ is bounded below and $\inf A + \inf B = \inf C$.*

Some properties of upper and lower bounds

Proof.

We prove only the first one, because the second one is analogous.
By the completeness axiom, A and B have the supremum $\sup A, \sup B$. As $\sup A$ and $\sup B$ are upper bounds of A and B respectively, for any element $z \in C$ we have $x \in A, y \in B$ such that $z = x + y$ and $x \le \sup A$, $y \le \sup B$ hence $z = x + y \le \sup A + \sup B$. In particular, $\sup A + \sup B$ is an upper bound of C, hence $\sup C \le \sup A + \sup B$.
Conversely, we know that, for any $\epsilon > 0$, there is $x \in A$ (and $y \in B$) such that $\sup A - \frac{\epsilon}{2} < x$ (and $\sup B - \frac{\epsilon}{2} < y$). Therefore, $\sup A + \sup B - \frac{\epsilon}{2} - \frac{\epsilon}{2} = \sup A + \sup B - \epsilon < x + y \le \sup C$ for arbitrary $\epsilon > 0$, hence by Lemma, $\sup A + \sup B \le \sup C$. Altogether, hence $\sup C = \sup A + \sup B$.

□

Some properties of upper and lower bounds

Remember that $\sup A$ is the least (smallest) upper bound and $\inf B$ is the greatest (largest) lower bound.

Theorem

Let $A, B \subset \mathbb{R}$. If for any $x \in A$ and $y \in B$ it holds that $x < y$, then $\sup A \leq \inf B$.

Proof.

Any $y \in B$ is an upper bound of A, hence $\sup A \leq y$. This means that $\sup A$ is a lower bound of B, hence $\sup A \leq \inf B$. □

The square roots of real numbers

Theorem

For any $a \in \mathbb{R}, a > 0$, *there is* $s \in \mathbb{R}, s > 0$ *such that* $s^2 = a$.

We denote it by $s = \sqrt{a}$.
For any $n \in \mathbb{N}$, we can define the n-th root of any positive number a and we denote it by $a^{\frac{1}{n}}$. The existence can be proved similarly.

Proof.

Let $A = \{x \in \mathbb{R} : x^2 < a\} \subset \mathbb{R}$. Then A is bounded above: Indeed, as $x^2 < a$, there are two cases:

- if $a > 1$, then $x^2 < a^2$ and hence $x < a$.
- if $a \leq 1$, then $x^2 < 1$ and hence $x < 1$.

In either case, A is bounded.

Let $s = \sup A \in \mathbb{R}$. We prove $s^2 = a$ by contradiction.

- if $s^2 < a$, then we take $\epsilon > 0$ such that $0 < \epsilon < \frac{a-s^2}{s}$ (or $s\epsilon < a - s^2$) and $\epsilon < s$. Then $(s + \frac{\epsilon}{4})^2 = s^2 + s\frac{\epsilon}{2} + \frac{\epsilon^2}{16} < s^2 + \frac{s\epsilon}{2} + \frac{s\epsilon}{2} < s^2 + s\epsilon < a$, therefore, s is not an upper bound of A (because $s + \frac{\epsilon}{4} \in A$), contradiction.
- if $s^2 > 2$, then we take $\epsilon > 0$ such that $0 < \epsilon < \frac{s^2-a}{s}$ (or $s\epsilon < s^2 - a$) and $\epsilon < s$. Then $(s - \frac{\epsilon}{4})^2 = s^2 - s\frac{\epsilon}{2} + \frac{\epsilon^2}{16} > s^2 - s\epsilon > a$, therefore, s is not the least upper bound of A (because $s - \frac{\epsilon}{4} \in A$ is another upper bound, smaller than s), contradiction.

$\square$

Decimal representations of real numbers

We denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
Any (positive) real number $x \in \mathbb{R}$ can be written in the form $x = a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \cdots$, where a_0 is an integer and $a_1, a_2, \cdots$ are integers between 0 and 9 (negative numbers can be most commonly written as $-\sqrt{2} = -1.41421\cdots$, although an analogous representation can apply to negative numbers).
Examples:

- $\frac{1}{3} = 0.33333\cdots$
- $\sqrt{2} = 1.41421\cdots$
- $\pi = 3.14159\cdots$

Decimal representations of real numbers

Indeed, let $x \in \mathbb{R}$ be a real number and $x > 0$. By the Archimedean property, there is a natural number $n \in \mathbb{N}_0$ such that $n - 1 \leq x < n$ (this is possible, because any subset of $\mathbb{N}$ has the minimal element, which we prove below). We take $a_0 = n - 1$.
Note that $0 < x - a_0 < 1$. Therefore, $0 < 10(x - a_0) < 10$. Take $a_1 \in \mathbb{N}_0$ the largest natural number such that $a_1 \leq 10(x - a_0)$. As it is the largest, we have again $0 < 10(x - a_0) - a_1 < 1$.
We can repeat this procedure and obtain a_n, and it always hold that
$x - a_0.a_1 \cdots a_n < 0.\underbrace{0 \cdots 01}_{n\text{-digits}}$.

Let $A = \{a_0, a_0.a_1, a_0.a_1a_2, a_0.a_1a_2a_3, \cdots\}$. This A is bounded (by $a_0 + 1$), hence it has the supremum s. Note that x is un upper bound of A, hence $\sup A \leq x$. On the other hand, if for any $\epsilon = 0.\underbrace{0 \cdots 01}_{n\text{-digits}}$, we have

$x - \epsilon < a_0.a_1 \cdots a_n \in A$, therefore, $x \leq \sup A$. Altogether, $x = \sup A = s$.

Decimal representations of real numbers

Theorem

A real number that is has nonrepeating decimal representation is irrational.

Proof.

We prove that any (positive) rational number has a repeating decimal representation. Then the claim follows by contradiction.
Let $x = a_0.a_1a_2\cdots = \frac{p}{q}$, $p, q \in \mathbb{N}$. We can write $p = nq + r$, where $n, r \in \mathbb{N}$ and $0 \le r < q$ (division with remainder). We set $a_0 = n$. Then we write $10r = n_1q + r_1$ again, and wet $a_1 = n_1$. In this way, we obtain the decimal representation of $\frac{p}{q}$, but there are only finitely many possible values $0, 1, \cdots q-1$ of r_1 because we are doing the division with remainder by q. This means that the numbers repeat after at largest q digits. □

The converse of this (any irrational number has a nonrepeating decimal representation) will be proven later.

Exercises

- Draw the set on the line $[-1,2] \cup (-3,-2) \cup (0,5]$.
- Determine the inf and sup of $[-4,0) \cup (2,3)$.
- Determine the set $(1,3) + (-2,2]$.
- Determine the set $5 \cdot (2,3)$.
- Represent the set $\{x \in \mathbb{R} : x^2 - 2x + 1 < 0\}$ as an interval.
- Represent the set $\{x \in \mathbb{R} : x^2 - 5x + 6 > 0\}$ as a union of intervals.
- Determine the decimal representation of $\frac{3}{7}$.
- Give an algorithm to produce a nonrepeating decimal representation.