Mathematical Analysis I: Lecture 3

Lecturer: Yoh Tanimoto

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Start recording...

Some tips

- Writing math.
  - LaTeX. You can try it here, and you can install the full set afterwards. You need to learn some commands, but once you know it it's very powerful. All my lecture notes and slides are written in LaTeX
  - Word processor (MS Word, Apple Pages, Open Office, Libre Office (Insert $\to$ Objects $\to$ Formula)...).
- Doing quick computations.
  - Wolfram Math Alpha You can just type some formulas in and it shows the result.
  - Programming languages. Python (I used it to make the graph of the SIR model), Java, C,$\cdots$
- How many of you are in Rome or plan to come to Rome soon? Please take this survey (or click the link in the chat).
- Today: Apostolo Vol. 1, Chapter I.3.

Are rational numbers all we need?

It is true that, in the real world, we can measure quantities to a certain accuracy, so we get numbers in a decimal representation:

- $c = 299792458[\mathrm{m} \cdot \mathrm{s}^{-1}]$ (the speed of light)
- $G = 0.0000000000667430(15)[\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}]$ (the gravitational constant), where (15) means these digits might be incorrect.
- Any other measured quantity in the real world.

And any experiment has a certain accuracy, so it makes sense only to write a certain number of digits, so rational numbers seem to suffice.

Are rational numbers all we need?

But for certain cases, we know that we should consider **irrational numbers**. For example,

- $\sqrt{2} = 1.41421356\cdots$, the number x such that $x^2 = 2$.
- $\pi = 3.1415926535\cdots$, the circumference of the circle with diameter 1.
- $e = 2.718281828\cdots$, Napier's number (we will define it in the lecture).
- Any decimal number which is not repeating.

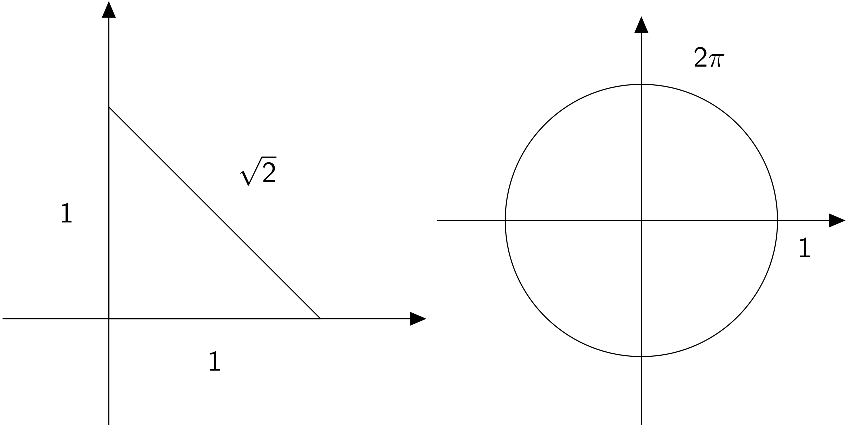

Figure: Left: the right triangle with equal sides 1. By the Pytagoras' theorem, the longest side is $\sqrt{1^2+1^2} = \sqrt{2}$. Right: the unit circle with radius 1 (diamenter 2). The length of the circle is 2π.

Irrationality of $\sqrt{2}$

For the next theorem, we need a **proof by contradiction**: by assuming the converse of the conclusion, we derive a contradiction, then we can conclude that the converse of the conclusion is false, that is, the conclusion is correct.
Recall that an integer p is **even** if it is a multiple of 2 (there is another integer r such that $p = 2r$), and p is **odd** if it is not even.

Theorem

$\sqrt{2}$ *is not a rational number.*

Irrationality of $\sqrt{2}$

Proof.

We prove this by contradiction, that is, we assume that $\sqrt{2}$ is a rational number. So there are integers p, q such that $\sqrt{2} = \frac{p}{q}$. We may assume that this is already reduced (that is, not a fraction like $\frac{4}{8}$ but like $\frac{1}{2}$. It's the form which you cannot simplify further).

As $\sqrt{2} \cdot \sqrt{2} = 2$, we have $\frac{p}{q} \cdot \frac{p}{q} = \frac{p^2}{q^2} = 2$, hence $p^2 = 2q^2$. As $\frac{p}{q}$ is reduced, there are two cases.

- if p is odd, then the equality $p^2 = 2q^2$ is even $=$ odd, contradiction.
- if p is even, then q is odd and we can write is as $p = 2r$, with another integer r, and $p^2 = 4r^2 = 2q^2$, and $2r^2 = q^2$. This is even $=$ odd, contradiction.

So, in all cases we arrived a contradiction from the assumption that $\sqrt{2}$ is rational. This means that $\sqrt{2}$ is irrational. $\square$

Irrational numbers

Exercise. Prove that $2\sqrt{2}$ is irrational.

It has been proven that π and e are irrational, but they are more difficult. Instead, it can be easily proven that any nonrepeating decimal number cannot be rational. This means *there are many irrational numbers*.
In other words, the set of rational numbers have "many spaces between them". We should fill them in with irrational numbers, so that the set of real numbers is a "continuum".

The axioms of the real numbers

Here we start the study of Mathematical Analysis, based on the set of **real numbers**. Our approach is synthetic, in the sense that we take the axioms for real numbers for granted, and develop the theory on them. It is also possible to "costruct" real numbers from rational numbers, and rational numbers from integers, integers from natural numbers, and so on, but at some point we have to assume certain axioms for simpler objects. If you are interested, look at "Dedekint's cut" (for real numbers), or "Peano's axioms" (for natural numbers).

The axioms of the real numbers

We assume that, the set $\mathbb{R}$ of **real numbers** is equipped with operations $+$, $\cdot$ and an order relation $<$ and for x, y, z real numbers, $x + y$ and $x \cdot y$ are again real numbers and they satisfy (*just the same for* $\mathbb{Q}$)

- (commutativity) $x + y = y + x, x \cdot y = y \cdot x$
- (associativity) $(x + y) + z = x + (y + z), (x \cdot y) \cdot z = x \cdot (y \cdot z)$
- (distributive law) $(x + y) \cdot z = xz + yz$
- (zero and unity) There are special distinct rational numbers, called 0 and 1, such that $x + 0 = x$ and $x \cdot 0 = 0$. And $x \cdot 1 = x$.
- (negative) There is a only one rational number, which we call $-x$, such that $x + (-x) = 0$.
- (inverse) If $x \neq 0$, there is only one rational number, which we call x^{-1}, such that $x \cdot x^{-1} = 1$.
- if $0 < x, 0 < y$, then $0 < xy$ and $0 < x + y$.
- if $x < y$, then $x + z < y + z$.
- if $x \neq 0$, either $0 < x$ or $x < 0$ but not both.
- $0 \not< 0$

The axioms of the real numbers

We can prove Theorems for real numbers corresponding to Theorems on rational numbers. Therefore, the real numbers have the same properties as the rational numbers, concerning the sum, product and order.

In addition, we will add an axiom about **completeness** (or continuum).

We say that $S \subset \mathbb{R}$ is **bounded above** if there is $x \in \mathbb{R}$ such that for any $y \in S$ it holds that $y \leq x$, and we write $S \leq x$. S is said to be **bounded below** if there is $x \in \mathbb{R}$ such that for any $y \in S$ it holds that $y \geq x$, and we write $S \geq x$.
If S is both bounded above and below, we say that S is **bounded**.

If S is bounded above, then any $x \in \mathbb{R}$ such that $S \leq x$ is called an **upper bound** of S. Similarly, if $x \leq S$, then x is said to be a **lower bound** of S.

If S has a least upper bound, that is there is x such that $S \leq x$ and $x \leq y$ for any upper bound y of S, then x is called the **supremum** of S and we denote it by $x = \sup S$. Similarly, if S has a largest lower bound x, then it is called the **infimum** of S and we denote it by $x = \inf S$.

$\mathbb{R}$ includes $\mathbb{Z}$ and $\mathbb{Q}$: $1 \in \mathbb{R}$, hence $2 = 1 + 1, 3 = 1 + 1 + 1, \cdots$ and $-1, -2, \cdots \in \mathbb{R}$. Also, if $p, q \in \mathbb{Z}$, $\frac{p}{q} \in \mathbb{R}$.

The completeness axioms

What distinguishes $\mathbb{R}$ from $\mathbb{Q}$ is the following.

- (the least upper bound axiom, or the completeness axiom) every nonempty subset S of $\mathbb{R}$ which is bounded above has a supremum: there is $B \in \mathbb{R}$ such that $B = \sup S$.

This should imply that $\sqrt{2} = 1.41421356\cdots$ belongs to $\mathbb{R}$! Indeed, let us take, by chopping the digits of $\sqrt{2}$, $S = \{1, 1.4, 1.41, 1.414, 1.4142, \cdots\}$. S is bounded above, indeed, $1.5 > 1, 1.4, 1.41, 1.414, \cdots$. On the other hand, if x has a decimal representation, e.g. 1.415, then there is a smaller number $x' = 1.4149$. So, $\sup S$ should be exactly $\sqrt{2}$. We will see this more precisely later.

Figure: The set S approximating $\sqrt{2}$, which is bounded by 1.5.

The Archimedean property

(A lemma is a theorem (a consequence of axioms) used to prove a more important theorem)

Lemma

If $S \subset \mathbb{R}$ *is bounded above and* $B = \sup S$*, then for any* $\epsilon > 0$*, there is* $x \in S$ *such that* $B - \epsilon < x$*.*

Proof.

By contradiction, assume that there is $\epsilon > 0$ such that $B - \epsilon \geq x$ for all $x \in S$. Then B is not the *least* upper bound, because $B - \epsilon$ is an upper bound of S and $B - \epsilon < B$. □

The Archimedean property

Theorem

The set $\mathbb{N} = \{1, 2, 3, \cdots\}$ *is not bounded above.*

Proof.

By contradiction, assume that $\mathbb{N}$ were bounded above. Then by the completeness axiom, there is $x = \sup \mathbb{N}$. By the lemma above, for $\epsilon = \frac{1}{2}$, there is $n \in \mathbb{N}$ such that $x - \frac{1}{2} < n$. But then $x < n + \frac{1}{2} < n + 1 \in \mathbb{N}$, and this contradicts the assumption that x were the upper bound of $\mathbb{N}$. This implies that $\mathbb{N}$ is not bounded above. $\square$

The Archimedean property

(A corollary is a theorem which follows easily from a more complicated theorem)

Corollary

For any $x \in \mathbb{R}$, there is $n \in \mathbb{N}$ such that $x < n$. For any $y, z \in \mathbb{R}$ and $z > y$, there is $n \in \mathbb{N}$ such that $\frac{1}{n} < z - y$.

Proof.

By the theorem above, x is not an upper bound of $\mathbb{N}$, so there is n such that $x < n$. By applying this to $\frac{1}{z-y}$, there is n such that $\frac{1}{z-y} < n$, which implies that $\frac{1}{n} < z - y$. $\square$

The graphical representation of $\mathbb{R}$

Therefore, we can represent the set of real numbers by a straight line, and any point $x \in \mathbb{R}$ is on the line and it falls between an integer n and another $n-1$ (possibly $x = n$). Conversely, any point on the line gives an element in $\mathbb{R}$.

Any real number $\mathbb{R}$ has a decimal representation (next lecture).

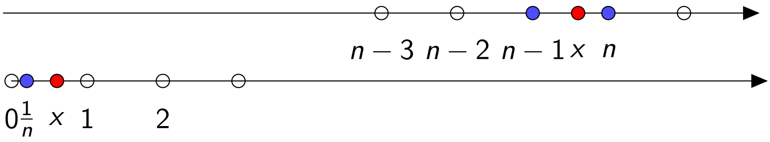

Figure: Any $x \in \mathbb{R}$ falls between $n-1$ and n (including equality) for some $n \in \mathbb{N}$. For any $x > 0$, there is $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

Note that $\mathbb{Q}$ does not have the completeness property!

Theorem

Let $A = \{x \in \mathbb{Q} : x^2 < 2\} (\subset \mathbb{R})$. *Then* A *is bounded above but* $\sup A \notin \mathbb{Q}$.

Proof.

A is bounded above, indeed, if $x^2 < 2$, then $x^2 < 4 = 2^2$, and hence $x < 2$. Let $s = \sup A \in \mathbb{R}$. Then $s^2 = 2$. We prove this by contradiction.

- if $s^2 < 2$, then we take $\epsilon > 0$ such that $0 < \epsilon < \frac{2-s^2}{s}$ (or $s\epsilon < 2 - s^2$) and $\epsilon < s$. Then $(s + \frac{\epsilon}{4})^2 = s^2 + s\frac{\epsilon}{2} + \frac{\epsilon^2}{16} < s^2 + \frac{s\epsilon}{2} + \frac{s\epsilon}{2} < s^2 + s\epsilon < 2$, therefore, s is not an upper bound of A (because $s + \frac{\epsilon}{4} \in A$), contradiction.
- if $s^2 > 2$, then we take $\epsilon > 0$ such that $0 < \epsilon < \frac{s^2-2}{s}$ (or $s\epsilon < s^2 - 2$) and $\epsilon < s$. Then $(s - \frac{\epsilon}{4})^2 = s^2 - s\frac{\epsilon}{2} + \frac{\epsilon^2}{16} > s^2 - s\epsilon > 2$, therefore, s is not the least upper bound of A (because $s - \frac{\epsilon}{4} \in A$ is another upper bound, smaller than s), contradiction.

But $s \in \mathbb{Q}$ never satisfies $s^2 = 2$. Hence $s = \sup A (= \sqrt{2}) \notin \mathbb{Q}$. $\square$

Exercises

- Prove that $2\sqrt{2}$ is irrational.
- Prove that $\sqrt{3}$ is irrational.
- Let $A = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\} = \{\frac{1}{n} : n \in \mathbb{N}\}$. Determine $\inf A$ and $\sup A$.
- Let $A = \{0.9, 0.99, 0.999, \cdots\}$. Determine $\inf A$ and $\sup A$.
- Let $A = \{0.3, 0.33, 0.333, \cdots\}$. Determine $\inf A$ and $\sup A$.
- $x = 0.000001$. For which n does it hold that $\frac{1}{n} < x$?
- Draw the graph of the set $\{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x\}$.
- Draw the graph of the set $\{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2\}$.
- Draw the graph of the set $\{(x, y) \in \mathbb{R} \times \mathbb{R} : y < x^2 + 1\}$.