Mathematical Analysis I: Lecture 2

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Start recording...

Some annoucements

- **Slides, lecture notes, exercises**: Team → Generale → Course page (tab on the top).
- **Recorded videos**: Team → Lectures → Recorded lectures (tab on the top).
- Office hours: Tuesday 10:00 – 11:00
- No lecture tomorrow (September 24th)
- A make-up lecture for September 24th: September 29th (Tuesday) 11:30–13:00, we do mostly exercises.
- Ask questions! But during the lecture I might not see your comments in chat. It's better if you use the microphone. At the end of lecture, I try to read and answer questions in chat.
- How many of you are in Rome or plan to come to Rome soon? Please take this survey.

Order in rational numbers

In the last lecture we learned the rational numbers and $+, \cdot$.
There is also an **order relation** $<$ ("x is larger than y": $y < x$) which satisfies, for x, y, z rational,

- if $0 < x, 0 < y$, then $0 < xy$ and $0 < x + y$.
- if $x < y$, then $x + z < y + z$.
- if $x \neq 0$, either $0 < x$ or $x < 0$ but not both.
- $0 \not< 0$ (meaning that it is not true that $0 < 0$)

$x < y$ and $y > x$ have the same meaning.
We say that x is **positive** if $0 < x$ and x is **negative** if $x < 0$. If x is not positive, then either $x = 0$ or $x < 0$ and in this case we say x is **nonpositive** and write $x \leq 0$ (again, $x \leq 0$ and $0 \geq x$ mean the same thing). Similarly, if $x > 0$ or $x = 0$, we say x is **nonnegative** and write $x \geq 0$.

Order in rational numbers

In addition to the "axioms", we also use the logic that, if an equality or inequality holds for some x and if $x = y$, then it also holds for y.
Example: if $x < z$ and $x = y$, then $y < z$.

Notation: for any number x, we write $x^2 = x \cdot x$. Similarly, $x^3 = x \cdot x \cdot x$, and so on.

Order in rational numbers

With the properties above, we can prove the following.

Theorem

Let a, b, c, d rational numbers. Then

- $a < b$ *if and only if* $a - b < 0$.
- *one and only one of the following holds:* $a < b, b < a, a = b$.
- *if* $a < b, b < c$ *then* $a < c$.
- *if* $a < b$ *and* $c > 0$, *then* $ac < bc$.
- *if* $a \neq 0$, *then* $a^2 (= a \cdot a) > 0$.
- $1 > 0$.
- *if* $a < b$ *and* $c < 0$, *then* $ac > bc$.
- *if* $a < b$, *then* $-a > -b$.
- *if* $ab > 0$, *then either* $a > 0, b > 0$ *or* $a < 0, b < 0$.
- *if* $a < c$ *and* $b < d$, *then* $a + b < c + d$.

Proof.

We only prove a few of them and leave the rest as exercises.

- If $a < b$, then by adding $-b$ to both sides, we get $a - b < 0$. Conversely, if $a - b < 0$, by adding b to both sides we get $a < b$.
- If $a = b$, then $b - a = 0$ and we know that both $b - a > 0$ and $b - a < 0$ are false and hence $b > a$ and $b < a$ are false.
- If $a < b$, then $a - b < 0$ and $a - b = 0$ is false, and hence $a = b$ is false.
- If $a < b$, then $0 < b - a$ and $0 < c \cdot (b - a) = bc - ac$, hence $ac < bc$.
- If $a \neq 0$, then either $a > 0$ or $a < 0$. For the case $a > 0$, we have $a^2 = a \cdot a > 0$. For the case $a < 0$, we have $-a > 0$ and $a^2 = (-a)^2 > 0$.

□

Order in rational numbers

All these "theorems" about rational numbers should be well-known to you. But it is important that we could prove them from a few axioms, which we assume to be true.

Exercises: Check the remaining statements.

Naive set theory

It is very often convenient to consider sets of numbers. For example, we may consider the set $\mathbb{Q}_+$ of positive rational numbers, or the set of multiples of 2, and so on. In mathematics, a **set** is a collection of mathematical objects.

The most precise treatment of sets requires a theory called axiomatic set theory, but in this lecture we think of a set simply as a collection of known objects.

Naive set theory

We often use capital letters $A, B, C, \cdots$ for sets, but in any case we declare that a symbol is a set. For a set S, we denote by $x \in S$ the statement "x is an **element** of S". We have already seen examples of sets: let us give them special symbols.

- $\mathbb{Q}$: the set of rational numbers
- $\mathbb{Z}$: the set of integers

Naive set theory

In general, we can consider two ways of constructing sets.

- By nomination. We can nominate all elements of a set. For example, $A = \{0, 1, 2, 3\}$ and $B = \{1, 10, 100, 1000\}$ are sets.
- By specification. We include all elements of an existing set with specific properties. For example, $A = \{x \in \mathbb{Z} : \text{ there is } y \in \mathbb{Z} \text{ such that } x = 2y\}$ (read that "A is the set of integers such that there is an integer y such that $x = 2y$") is the set of multiples of 2 (we recycled the symbol A. When we do this, we shall always declare it).

For a set constructed by nomination, the order and repetition do not matter: $\{0, 1, 2, 3\} = \{0, 3, 2, 1\} = \{0, 0, 1, 1, 1, 2, 3\}$. In other words, a set is defined by its elements.

Naive set theory

A construction by specification appears very often. Let us introduce a more symbol.

- $\mathbb{N} = \{x \in \mathbb{N} : x > 0\}$ is called the set of **natural numbers**.
- $\emptyset$ is the set that contains nothing and called the **empty set**. $\emptyset$ is a subset of any set: if A is a set, the statement "if $x \in \emptyset$ then $x \in A$" is satisified just because there is no such x!

Subsets

Let B be a set. We say that A is a **subset** of B if all elements of A belong to B, and denote this by $A \subset B$. It holds that $A \subset A$ for any set A.

Example

- Let $A = \{1, 2, 3\}$ and $B = \{0, 1, 2, 3, 4\}$. Then $A \subset B$.
- $\mathbb{N} \subset \mathbb{Z}$.
- Let $A = \{1, 2, 3, 4, 5, 6\}$. Then $A \subset \mathbb{N}$.

It may happen that $A \subset B$ and $B \subset A$, that is, all elements of A belong to B and vice versa. This means that A and B are the same as sets, and in this case we write $A = B$.

Subsets

The definition by specification
$A = \{x \in B : x \text{ satisfies the property XXX...}\}$ gives always a subset, in this case of B. Note also that x in this definition has no meaning ("dummy"). One can write it equivalently
$A = \{y \in B : y \text{ satisfies the property XXX...}\}$.
For $x \in A$, the set $\{x\}$ that contains only x should be distinguished from x. It is a subset of A: $\{x\} \subset A$.

If A and B are sets, then we can consider the set which contains the elements of A and B, and nothing else. It is called the **union** of A and B and denoted by $A \cup B$.

Example

- Let $A = \{1, 2, 3\}$ and $B = \{0, 1, 3, 4\}$. Then $A \cup B = \{0, 1, 2, 3, 4\}$.

Unions, intersections, complements

Similarly, we can consider the set of all the elements which belong both to A and B, and nothing else. It is called the **intersection** of A and B and denoted by $A \cap B$.

Example

- Let $A = \{1, 2, 3\}$ and $B = \{0, 1, 3, 4\}$. Then $A \cap B = \{1, 3\}$.

Furthermore, the **difference** of B with respect to A is all the element of A that do not belong to B and is denoted by $A \setminus B$ (note that this is different from $B \setminus A$).

Example

- Let $A = \{1, 2, 3\}$ and $B = \{0, 1, 3, 4\}$. Then $A \setminus B = \{2\}$.

Unions, intersections, complements

We can consider the union of more than two sets: $A \cup (B \cup C)$. By considering the meaning, this set contains all the elements which belong either A or $B \cup C$, which is to say all elements which belong either A or B or C. Therefore, the order does not matter and we can write $A \cup B \cup C$.
Similarly, $A \cap B \cap C$ is the intersection of A, B and C.
We may consider a **family of sets** $\{A_i\}_{i \in I}$ **indexed by another set** I. For example, we can take $\mathbb{N}$ as the index set and
$A_n = \{m \in \mathbb{Z} : m \text{ is a multiple of } n\}$. For a family of set, we can define the union and the intersection analogously and we denote them by

$$\bigcup_{i \in I} A_i, \qquad \bigcap_{i \in I} A_i,$$

respectively.

The set of subsets

We can consider also certain **sets of sets**.

Example

- $\{1,2,3\}, \{2\}, \{1,4,6,7\}$ are sets. We can collect them together

$$\{\{1,2,3\}, \{0,2\}, \{1,4,6,7\}\}.$$

  This is a set of sets.

- Let $A = \{1,2,3\}$. We can collect all subsets of A:

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

- One can also consider the set of all subsets of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, but we cannot name all the elements: they are infinite.
  For example, for $\mathbb{N} = \{1,2,3,4,\cdots\}$, the set of subsets of $\mathbb{N}$ is infinite.

Ordered pairs

For sets A, B, we can consider **ordered pairs** of elements in A and B.

Example

- Let $A = \{1, 2, 3\}, B = \{3, 4\}$. Then the sef $A \times B$ of the ordered pairs of A, B is

$$A \times B := \{(1, 3), (2, 3), (3, 3), (1, 4), (2, 4), (3, 4)\}.$$

- If we take $\mathbb{N}$, then $\mathbb{N} \times \mathbb{N}$ is the set of all ordered pairs of natural numbers. $\mathbb{N} \times \mathbb{N} = \{(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), \cdots\}$.

Ordered pairs

Ordered pairs can be described using **graphs**. If $A, B \subset \mathbb{Z}$ have finitely many points, say m, n respectively, then there are $m \cdot n$ ordered pairs. We take the horizontal axis for A and the vertical axis for B.
To obtain the graph of $A \times B$, we should mark the point (x, y) if and only if $x \in A$ and $y \in B$. For any subset X of $A \times B$, we should mark the point (x, y) if and only if $(x, y) \in X$.

Ordered pairs

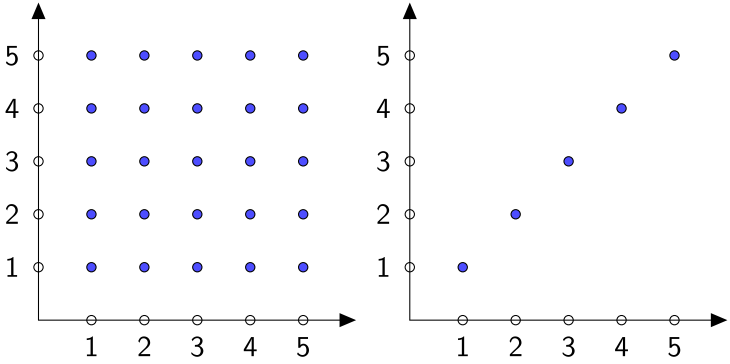

Figure: Left: the set of all ordered pairs $\{1,2,3,4,5\} \times \{1,2,3,4,5\}$. Right: a subset $\{(1,1),(2,2),(3,3),(4,4),(5,5)\} \subset \{1,2,3,4,5\} \times \{1,2,3,4,5\}$.

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Exercises

- For rational x, y, z, prove that if $x < y$ and $y < z$, then $x < z$.
- Let $A = \{0, 1, 2, 3\}, B = \{x \in \mathbb{Z} : \text{ there is } y \in \mathbb{Z} \text{ such that } x = 2y\}$. What are $A \cap B$ and $A \cup B$?
- Let $A = \{x \in \mathbb{Z} : \text{ there is } y \in \mathbb{Z} \text{ such that } x = 3y\}, B = \{x \in \mathbb{Z} : \text{ there is } y \in \mathbb{Z} \text{ such that } x = 2y\}$. What are $A \cap B$ and $A \cup B$?
- Let $A_n = \{x \in \mathbb{Z} : \text{ there is } y \in \mathbb{Z} \text{ such that } x = ny\}$. What are $\bigcap_n A_n$ and $\bigcup_n A_n$?
- Let $A = \{1, 2, 3, 4, 5, 6\}$, and $B = \{(x, y) \in A \times A : y = 2x\}$. Give all elements of B and draw its graph.
- Let $A = \mathbb{Z}$, and $B = \{(x, y) \in A \times A : y > x + 2\}$. Give all elements of B and draw (a part of) its graph. What if $A = \mathbb{Q}$?