BSc Engineering Sciences – A. Y. 2018/19

Written exam of the course Mathematical Analysis 2

June 24, 2019

Last name: .... First name: ....
Matriculation: ....

Solve the following problems, motivating in detail the answers.

1.

(1) Find a function $g : \mathbb{R} \to \mathbb{R}$ such that the solution $u(x,t)$ of the wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = \dfrac{1}{1+x^2} \\ \dfrac{\partial u}{\partial t}(x,0) = g(x) \end{cases}$$

is given by $u(x,t) = \frac{1}{1+(x-ct)^2}$ (Note after the exam: in the original paper there was a typo in the last line: $\frac{\partial u}{\partial x}$ should have been $\frac{\partial u}{\partial t}$).

(2) Find a function $g : (0,+\infty) \to \mathbb{R}$ such that $f(x,y) = g(\sqrt{x^2+y^2})$ is a solution of the 2-dimensional Poisson equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \sqrt{x^2+y^2}$$

(Hint: express the Laplacian $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ in polar coordinates.)

Solution.

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2.

(1) Let us set $(x_1, y_1) = (1, 2), (x_2, y_2) = (2, 3), (x_3, y_3) = (-1, 1)$. Find and classify all the stationary points $(a, b) \in \mathbb{R}^2$ of the function $f(a, b) = \sum_{n=1}^{3}(ax_n + b - y_n)^2$.

(2) Compute the derivative of the following function $g(x)$ of x:

$$g(x) = \int_{-\sin x}^{1} \cos(t^3)dt.$$

Solution.

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3. Determine whether the following vector field on $\mathbb{R}^2$

$$\boldsymbol{f}(x,y) = \big(\cos xy - xy\sin xy,\ -x^2\sin xy + x^3\big)$$

is a gradient of some scalar field. Depending on this result,

- If $\boldsymbol{f}(x,y)$ is a gradient, find one of these scalar fields φ such that $\boldsymbol{f}(x,y) = \nabla\varphi(x,y)$.
- If $\boldsymbol{f}(x,y)$ is not a gradient, compute $\int_C \boldsymbol{f}\cdot d\boldsymbol{\alpha}$, where

$$\boldsymbol{\alpha}(t) = \begin{cases} (t,0) & 0\le t\le 1 \\ (1-(t-1),(t-1)) & 1\le t\le 2 \\ (0,1-(t-2)) & 2\le t\le 3 \end{cases}$$

Solution.

Matriculation: ....

4. Compute the integral

$$\iiint_D (\sin x + y^2) z \, dx dy dz$$

where

$$D = \left\{ (x, y, z) \in \mathbb{R}^3 \, : \, \sqrt{\frac{x^2}{9} + \frac{y^2}{4}} \leq z \leq \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \right\}.$$

Solution.

Matriculation: ....

5. Let $\boldsymbol{f}(x, y, z) = (xz, yz, 0)$ be a vector field on $\mathbb{R}^3$ and

$$S = \{(x, y, z) : x^2 + y^2 - z^2 = -4,\ 0 \leq z \leq 3\}$$

be a surface in $\mathbb{R}^3$. Compute the surface integral

$$\iint\limits_S \boldsymbol{F} \cdot \boldsymbol{n} \, dS,$$

where $\boldsymbol{n}$ is a unit normal vector on S with positive z-component.
Solution.