BSc Engineering Sciences – A. Y. 2018/19

Written exam of the course Mathematical Analysis 2

January 29, 2019

Last name: .... First name: ....

Matriculation: ....

Solve the following problems, motivating in detail the answers.

1. (6 points) Find a power series solution $y(x)$ of the differential equation

$$y''(x) + 2x\,y'(x) - 8y(x) = 0,$$

such that $y(0) = 1$, $y'(0) = -1$, and determine its radius of convergence.

Solution. Let us put $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then we have $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$. If $y(x)$ satisfies the above equation, then

$$\begin{aligned}
0 &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2x \sum_{n=1}^{\infty} n a_n x^{n-1} - 8 \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 8a_n x^n \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 8a_n x^n \\
&= \sum_{n=0}^{\infty} \left((n+2)(n+1)a_{n+2} + 2na_n - 8a_n\right) x^n,
\end{aligned}$$

where in the second equality we shifted the index by $n \to n+2$ in the first summation, and in the third equality we used that the term $n = 0$ is 0 in the second summation. If this equality holds as power series, then it follows that $((n+2)(n+1)a_{n+2} + 2na_n - 8a_n) = 0$, or equivalently $a_{n+2} = -\frac{2(n-4)a_n}{(n+2)(n+1)}$.

From the initial conditions $y(0) = 1, y'(0) = -1$, we have $a_0 = 1, a_1 = -1$. By the recursion relation above, we obtain $a_2 = 4a_0 = 4, a_3 = a_1 = -1, a_4 = \frac{1}{3}a_2 = \frac{4}{3}, a_5 = \frac{a_3}{10} = -\frac{1}{10}$, and $a_6 = 0$. Furthermore, $a_{2n} = 0$ for $n \geq 3$ again by the recursion relation. By solving the recursion relation, for $n \geq 3$ we have $a_{2n+1} = (-1)^n 2^{n-2}(2n-5)!! \cdot \frac{5!a_5}{(2n+1)!} = (-1)^{n+1}2^n(2n-5)!! \cdot \frac{3}{(2n+1)!}$. With this a_n, $y(x) = \sum_{n=0}^{\infty} a_n x^n$ satisfies the given equation in the radius of convergence.

With $b_n = (-1)^{n-1}2^n(2n-5)!! \cdot \frac{3}{(2n+1)!}$, by ratio test we have $\frac{|b_{n+1}|}{|b_n|} = \frac{2(2n-3)}{(2n+3)(2n+2)} \to 0$, so the power series $\sum_{n=3}^{\infty} b_n y^n$ has the radius of convergence ∞. Since $\sum_{n=0}^{\infty} a_n x^n = 1 - x + 4x^2 - x^3 + \frac{4}{3}x^4 - \frac{1}{10}x^5 + x\sum_{n=3}^{\infty} b_n (x^2)^n$, the former has the same radius of convergence ∞.

Matriculation: ....

2.

(1) (4 points)Find all the stationary points of the following scalar field, defined on $\mathbb{R}^2$,

$$f(x,y) = x^3 + y^3 - 6xy$$

and classify them into relative minima, maxima and saddle points.

(2) (2 points) Compute the derivative of the following function of x:

$$g(x) = \int_{-x^2}^{x^2} e^{t^2} dt.$$

Solution.
(1) For the f given above, it holds that

$$\nabla f(x,y) = (3x^2 - 6y, 3y^2 - 6x).$$

At stationary points, $\nabla f(x,y) = \mathbf{0}$ holds. Namely,

$$3x^2 - 6y = 0, 3y^2 - 6x = 0.$$

By subtracting the both sides, one obtains $3(x^2-y^2)+6(x-y) = 3(x-y)(x+y)+6(x-y) = 0$.

- Consider the case $x - y = 0$. Then, from one of the equations above, $3x^2 - 6x = 3x(x-2) = 0$, therefore, $x = 0, 3$ and correspondingly, $(x,y) = (0,0), (2,2)$.
- Consider the case $x - y \neq 0$. Then we can divide the last equation by $x - y$ to obtain $3(x + y) + 6 = 0$, or equivalently, $y = -x - 2$. By substituting this in one of the equations above, $3x^2 - 6(-x-2) = 3(x^2 + 2x + 2) = 0$. It is easy to see that there is no real x that satisfies this equation.

To classify these points, let us compute the Hessian matrix:

$$\begin{pmatrix} 6x & -6 \\ -6 & 6y \end{pmatrix}.$$

- At the point $(x,y) = (0,0)$, this becomes

$$\begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix}.$$

Its determinant is -36, therefore, it has both negative and positive eigenvalues, and the point $(0,0)$ is a saddle point.

- At the point $(x,y) = (2,2)$, this becomes

$$\begin{pmatrix} 12 & -6 \\ -6 & 12 \end{pmatrix}.$$

Its determinant is 108 and its trace is 24, therefore, its eigenvalues are both positive, and the point $(0,0)$ is a relative minumum.

(2) Let us put $F(t)$ a primitive function of e^{t^2}, namely $F'(t) = e^{t^2}$, then $g(x) = F(x^2) - F(-x^2)$. By the chain rule, $g'(x) = 2x \cdot F'(x^2) - (-2x)F'(-x^2) = 4xe^{x^4}$.

Matriculation: ....

3. (6 points) Determine whether the following vector field on $\mathbb{R}^2$

$$\boldsymbol{f}(x,y) = \left(xe^{x^2+y^2} - y, ye^{x^2+y^2} + x\right)$$

is a gradient of some scalar field. Depending on this result,

- If $\boldsymbol{f}(x,y)$ is a gradient, find one of these scalar fields φ such that $\boldsymbol{f}(x,y) = \nabla\varphi(x,y)$.
- If $\boldsymbol{f}(x,y)$ is not a gradient, compute $\int_C \boldsymbol{f}\cdot d\boldsymbol{\alpha}$, where $\boldsymbol{\alpha}(t) = (\cos t, \sin t), t \in [0, 2\pi]$.

Solution. Let us call $\boldsymbol{f}(x,y) = (P(x,y), Q(x,y))$, where $P(x,y) = xe^{x^2+y^2} - y, Q(x,y) = ye^{x^2+y^2} + x$. We compute:

$$\frac{\partial P}{\partial y}(x,y) = 2xye^{x^2+y^2} - 1,$$
$$\frac{\partial Q}{\partial x}(x,y) = 2xye^{x^2+y^2} + 1,$$

and we see that they are different. This implies that $\boldsymbol{f}$ is not a gradient.

To compute the line integral $\int_C \boldsymbol{f}\cdot d\boldsymbol{\alpha}$, this is equal to $\iint_S \frac{\partial Q}{\partial y}(x,y) - \frac{\partial P}{\partial x}(x,y)dxdy$, by Green' theorem, where S is the region surrounded by the circle C, namely S is the disk $\{(x,y) : x^2+y^2 \le 1\}$, and we know that its area is π. Therefore,

$$\int_C \boldsymbol{f}\cdot d\boldsymbol{\alpha} = \iint_S \frac{\partial Q}{\partial y}(x,y) - \frac{\partial P}{\partial x}(x,y)dxdy = \iint_S 2dxdy = 2\pi.$$

Matriculation: ....

4. (6 points) Compute the integral

$$\iiint_D \frac{x+y}{\sqrt{2}} dxdydz,$$

where $D := \{(x,y,z) \in \mathbb{R}^3 \,:\, \frac{x^2+y^2}{4} + z^2 \le 1, y \ge 0\}$.
Solution. We can rewrite

$$\begin{aligned} D &= \{(x,y,z) \in \mathbb{R}^3 \,:\, \frac{x^2+y^2}{4} + z^2 \le 1,\ y \ge 0\} \\ &= \{(x,y,z) \in \mathbb{R}^3 \,:\, x^2+y^2 \le 4(1-z^2),\ 0 \le y \le \sqrt{4(1-z^2)},\ -1 \le z \le 1\} \\ &= \left\{ (x,y,z) \in \mathbb{R}^3 \,:\, \begin{array}{c} -1 \le z \le 1,\ 0 \le y \le \sqrt{4(1-z^2)}, \\ -\sqrt{4(1-z^2)-y^2} \le x \le \sqrt{4(1-z^2)-y^2} \end{array} \right\}. \end{aligned}$$

From the last expression we see that D is zy-projectable, so the triple integral in question can be written as an iterated integral. Furthermore, for each fixed z, the xy-integral becomes an integral on $D_0 = \{(x,y) \in \mathbb{R}^2 \,:\, x^2+y^2 \le 4(1-z^2),\ 0 \le y \le \sqrt{4(1-z^2)}\}$. Therefore,

$$\begin{aligned} \iiint_D \frac{x+y}{\sqrt{2}} dxdydz &= \int_{-1}^{1} \int_0^{\sqrt{4(1-z^2)}} \int_{-\sqrt{4(1-z^2)-y^2}}^{\sqrt{4(1-z^2)-y^2}} \frac{x+y}{\sqrt{2}} dxdydz \\ &= \int_{-1}^{1} \left[\iint_{D_0} \frac{x+y}{\sqrt{2}} dxdy \right] dz. \end{aligned}$$

Note that, going to the polar coordinates $x = r\cos\theta, y = r\sin\theta$, D_0 corresponds to the region

$$\tilde{D}_0 = \{(r,\theta) \in \mathbb{R}^2 : 0 \le r \le \sqrt{4(1-z^2)}, 0 \le \theta \le \pi\}.$$

Therefore, with the Jacobian $J(r,\theta) = r$, we obtain

$$\begin{aligned} \int_{-1}^{1} \left[\iint_{D_0} \frac{x+y}{\sqrt{2}} dxdy \right] dz &= \frac{1}{\sqrt{2}} \int_{-1}^{1} \left[\iint_{\tilde{D}_0} r(\cos\theta + \sin\theta) r dr d\theta \right] dz \\ &= \frac{1}{\sqrt{2}} \int_{-1}^{1} \left[\int_0^{\sqrt{4(1-z^2)}} \int_0^{\pi} r^2 (\cos\theta + \sin\theta) dr d\theta \right] dz \\ &= \frac{1}{\sqrt{2}} \int_{-1}^{1} \frac{16}{3} (1-z^2)^{\frac{3}{2}} dz. \end{aligned}$$

By the substitution $z = \sin t$, $\frac{dz}{dt} = \cos t$ and $1 - z^2 = \cos^2 t$,

$$\begin{aligned} \frac{1}{\sqrt{2}} \int_{-1}^{1} \frac{16}{3} (1-z^2)^{\frac{3}{2}} dz &= \frac{8\sqrt{2}}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 t dt = \frac{8\sqrt{2}}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2t}{2} \right)^2 dt \\ &= \frac{2\sqrt{2}}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + 2\cos 2t + \cos^2 2t) dt \\ &= \frac{2\sqrt{2}}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3}{2} + 2\cos 2t + \frac{\cos 4t}{2} \right) dt = \sqrt{2}\pi, \end{aligned}$$

where we used $\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$ twice and $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2t\, dt = 0 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 4t\, dt$.

Alternatively, one could use, in the original integral, cylindrical coordinates $x = r\cos\theta, y = r\sin\theta, z = z$, through which D is sent to

$$\begin{aligned} E &= \left\{(r,\theta,z) \in \mathbb{R}^3 \,:\, \frac{r^2}{4} + z^2 \leq 1, r\sin\theta \geq 0\right\} \\ &= \left\{(r,\theta,z) \in \mathbb{R}^3 \,:\, 0 \leq \theta \leq \pi, 0 \leq r \leq 2, -\sqrt{1-\frac{r^2}{4}} \leq z \leq \sqrt{1-\frac{r^2}{4}}\right\}, \end{aligned}$$

so that the given integral becomes

$$\frac{1}{\sqrt{2}} \int_0^\pi d\theta(\cos\theta + \sin\theta) \int_0^2 dr\, r^2 \int_{-\sqrt{1-\frac{r^2}{4}}}^{\sqrt{1-\frac{r^2}{4}}} dz = 2\sqrt{2}\int_0^2 dr\, r^2\sqrt{1-\frac{r^2}{4}},$$

and using the variable change $r = 2\sin t$,

$$\begin{aligned} 2\sqrt{2}\int_0^2 dr\, r^2\sqrt{1-\frac{r^2}{4}} &= 2\sqrt{2}\int_0^{\frac{\pi}{2}} dt\, 2\cos t(2\sin t)^2\cos t = 4\sqrt{2}\int_0^{\frac{\pi}{2}} dt\ \sin^2 2t \\ &= 4\sqrt{2}\int_0^{\frac{\pi}{2}} dt\ \left(\frac{1-\cos 4t}{2}\right) = \sqrt{2}\pi. \end{aligned}$$

Matriculation: ....

5. (6 points) Let $\boldsymbol{F}(x, y, z) = (x(x^2 + y^2 + z^2), y(x^2 + y^2 + z^2), z(x^2 + y^2 + z^2))$ be a vector field on $\mathbb{R}^3$, S be the surface of the sphere with radius $a > 0$:

$$S := \{(x, y, z) : x^2 + y^2 + z^2 = a^2\},$$

and $\boldsymbol{n}$ the outgoing normal unit vector on S at each point of S.

Compute the surface integral

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n} \, dS.$$

Solution. Thanks to Gauss' theorem (divergence theorem), this integral is equal to the following volume integral

$$\iiint_V \operatorname{div} \boldsymbol{F} \, dxdydz$$

where $V = \{(x, y, z) : x^2 + y^2 + z^2 \leq a^2\}$.

Let us compute:

$$\operatorname{div} \boldsymbol{F} = (x^2 + y^2 + z^2) + 2x^2 + (x^2 + y^2 + z^2) + 2y^2 + (x^2 + y^2 + z^2) + 2z^2 = 5(x^2 + y^2 + z^2).$$

To perform the volume integral, we use the spherical coordinate $x = r \cos\theta \sin\varphi, y = r \sin\theta \sin\varphi, z = r \cos\varphi$. The region Q correponding to V in this change of coordinate is $Q = \{(r, \theta, \varphi) : 0 \leq r \leq a,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \varphi \leq \pi\}$. Recall that the Jacobian determinant is $J(r, \theta, \varphi) = -r^2 \sin\varphi$, and note that $x^2 + y^2 + z^2 = r^2$. Theorefore,

$$\begin{aligned}
\iint_S \boldsymbol{F} \cdot \boldsymbol{n} \, dS &= \iiint_V \operatorname{div} \boldsymbol{F} \, dxdydz \\
&= 5 \iiint_Q r^2 \cdot r^2 \sin\varphi \, dr d\theta d\varphi \\
&= 5 \int_0^\pi \int_0^{2\pi} \int_0^a r^4 \sin\varphi \, dr d\theta d\varphi \\
&= 5 \cdot 2 \cdot 2\pi \cdot \frac{a^5}{5} \\
&= 4\pi a^5.
\end{aligned}$$