Mathematical Analysis II, 2018/19 First semester

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We basically follow the textbook "Calculus" Vol. I,II by Tom M. Apostol, Wiley.

Nov 26. Implicit functions and partial derivatives

1. The equation $x+z+(y+z)^2 = 6$ defines implicitly a function $f(x,y) = z$. Compute $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ in terms of x, y, z. Check that $(1,1,1)$ satisfies the equation, and compute $\frac{\partial f}{\partial x}(1,1), \frac{\partial f}{\partial y}(1,1)$.

   Solution. Put $F(x,y,z) = x+z+(y+z)^2 - 6$. The partial derivatives are $\frac{\partial F}{\partial x} = 1, \frac{\partial F}{\partial y} = 2(y+z), \frac{\partial F}{\partial z} = 1+2(y+z)$. For the partial derivaties of f, using the formula of the lecture,

$$\frac{\partial f}{\partial x}(x,y) = -\frac{\frac{\partial F}{\partial x}(x,y,f(x,y))}{\frac{\partial F}{\partial z}(x,y,f(x,y))} = -\frac{1}{1+2(y+f(x,y))},$$
$$\frac{\partial f}{\partial y}(x,y) = -\frac{\frac{\partial F}{\partial y}(x,y,f(x,y))}{\frac{\partial F}{\partial z}(x,y,f(x,y))} = -\frac{2(y+f(x,y))}{1+2(y+f(x,y))}.$$

   By putting $(x,y,f(x,y)) = (1,1,1)$,

$$\frac{\partial f}{\partial x}(1,1) = -\frac{1}{5},$$
$$\frac{\partial f}{\partial y}(1,1) = -\frac{\frac{\partial F}{\partial y}(x,y,f(x,y))}{\frac{\partial F}{\partial z}(x,y,f(x,y))} = -\frac{4}{5}.$$

2. Consider two surfaces $2x^2+3y^2-z^2-25 = 0$, $x^2+y^2-z^2 = 0$. The intersection C can be parametrized as $(X(z), Y(z), z)$.

   (a) Check that C passes the point $P = (\sqrt{7}, 3, 4)$.

   (b) Find a tangent vector of C at P.

   Solution.

   (a) By substituting $(x,y,z) = (\sqrt{7},3,4)$.

   (b) i. By implicit computations. Put $F(x,y,z) = 2x^2+3y^2-z^2-25, G(x,y,z) = x^2+y^2-z^2$. We use the formula $\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix}^{-1} \begin{pmatrix} -\frac{\partial F}{\partial z} \\ -\frac{\partial g}{\partial z} \end{pmatrix}$. These partial derivatives can be computed:

$$\frac{\partial F}{\partial x} = 4x, \frac{\partial F}{\partial y} = 6y, \frac{\partial F}{\partial z} = -2z, \frac{\partial G}{\partial x} = 2x, \frac{\partial G}{\partial y} = 2y, \frac{\partial G}{\partial z} = -2z.$$

By putting the value $P = (\sqrt{7}, 3, 4)$, we obtain

$$\begin{pmatrix} X'(4) \\ Y'(4) \end{pmatrix} = \begin{pmatrix} 4\sqrt{7} & 18 \\ 2\sqrt{7} & 6 \end{pmatrix}^{-1} \begin{pmatrix} -(-8) \\ -(-8) \end{pmatrix}$$

$$= \frac{1}{-12\sqrt{7}} \begin{pmatrix} 6 & -18 \\ -2\sqrt{7} & 4\sqrt{7} \end{pmatrix} \begin{pmatrix} 8 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{8}{\sqrt{7}} \\ -\frac{4}{3} \end{pmatrix}.$$

ii. By direct computations. We have, from $3G - F = 0, x^2 + 25 = 2z^2$, which is equivalent to $x = X(z) = \pm\sqrt{2z^2 - 25}$. Similarly, from $F - 2G = 0$, it follows that $y^2 = 25 - z^2$, which is equivalent to $y = Y(z) = \pm\sqrt{25 - z^2}$. Since we are interested in the point $(\sqrt{7}, 3, 4)$, we take the + solutions. By differentiating them, $X'(z) = \frac{2z}{\sqrt{2z^2-25}}, Y'(z) = -\frac{z}{\sqrt{25-z^2}}$. At P, $\begin{pmatrix} X'(4) \\ Y'(4) \end{pmatrix} = \begin{pmatrix} \frac{8}{\sqrt{7}} \\ -\frac{4}{3} \end{pmatrix}$.

Hence a tangent vector is $\begin{pmatrix} \frac{8}{\sqrt{7}} \\ -\frac{4}{3} \\ 1 \end{pmatrix}$.

Nov 26. Stationary points.

1. Locate and classify the stationary points.

   (a) $f(x, y) = x^2 + (y - 1)^2$

   (b) $f(x, y) = 2x^2 - xy - 3y^2 - 3x + 7y$

   (c) $f(x, y) = \sin x \cosh y$

   Solution.

   (a) $\nabla f(x, y) = (2x, 2(y - 1))$. $\nabla f(x, y) = \mathbf{0} \Leftrightarrow (x, y) = (0, 1)$. $H(x, y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, hence $H(0, 1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, and this has positive eigenvalues $2, 2$. Therefore, $(0, 1)$ is a local minumum.

   (b) $\nabla f(x, y) = (4x - y - 3, -x - 6y + 7)$. $\nabla f(x, y) = \mathbf{0} \Leftrightarrow (x, y) = (1, 1)$. $H(x, y) = \begin{pmatrix} 4 & -1 \\ -1 & -6 \end{pmatrix}$, hence $H(1, 1) = \begin{pmatrix} 4 & -1 \\ -1 & -6 \end{pmatrix}$, and this has positive and negative eigenvalues, because its determinant is -26. Therefore, $(1, 1)$ is a saddle point.

   (c) $\nabla f(x, y) = (\cos x \cosh y, \sin x \sinh y)$. $\nabla f(x, y) = \mathbf{0} \Leftrightarrow \cos x = 0$ and $\sinh y = 0 \Leftrightarrow (x, y) = \left((m + \frac{1}{2})\pi, 0\right)$. $H(x, y) = \begin{pmatrix} -\sin x \cosh y & \cos x \sinh y \\ \cos x \sinh y & \sin x \cosh y \end{pmatrix}$. Note that $\sin\left(m + \frac{1}{2}\right) \neq 0$. hence $H(\left(m + \frac{1}{2}\right), 0)$ has the determinant -1, and hence has positive and negative eigenvalues. Therefore, $(\left(m + \frac{1}{2}\right), 0)$ is a saddle point.

2. Let $x_1, \cdots, x_n$ be distinct numbers, $y_1, \cdots, y_n \in \mathbb{R}$. Let $a, b \in \mathbb{R}, f(x) = ax + b$. With $E(a, b) = \sum_{j=1}^n |f(x_j) - y_j|^2$. Find a, b which minimize $E(a, b)$.

   Solution. We can write

   $$E(a, b) = \sum_{j=1}^n (ax_j + b - y_j)^2.$$

   Therefore,

   $$\nabla E(a, b) = \left(\sum_{j=1}^n 2x_j(ax_j + b - y_j), \sum_{j=1}^n 2(ax_j + b - y_j) \right)$$

From $\nabla E(a,b) = \mathbf{0}$, we obtain

$$a\sum_{j=1}^n x_j^2 + b\sum_{j=1}^n x_j = \sum_{j=1}^n x_j y_j, \qquad a\sum_{j=1}^n x_j + b\sum_{j=1}^n = \sum_{j=1}^n y_j$$

Put $x^* = \frac{1}{n}\sum_{j=1}^n x_j, y^* = \frac{1}{n}\sum_{j=1}^n y_j$, then the second equation is $x^*a + b = y^*$, or $(x^*)^2 + x^*b = x^*y^*$. Set $u_j = x_j - x^*$, then the first equation is

$$\frac{a}{n}\sum_{j=1}^n x_j^2 + x^*b = \frac{1}{n}\sum_{j=1}^n x_j y_j.$$

By subtracting $(x^*)^2 + x^*b = x^*y^*$, we have

$$\frac{a}{n}\sum_{j=1}^n x_j u_j = \frac{1}{n}\sum_{j=1}^n u_j y_j,$$

hence by noting that $\sum_j u_j = 0$, $a = \sum_{j=1}^n u_j y_j / \sum_{j=1}^n x_j u_j = \sum_{j=1}^n u_j y_j / \sum_{j=1}^n u_j^2, b = y^* - x^*a$.

Nov. 9. Lagrange's multiplier method

1. Find the maximum and minimum distances from the origin to the curve $5x^2+6xy+5y^2 = 8$.
2. Assume $a, b \in \mathbb{R}, a, b > 0$.
   (a) Find the extreme values of $f(x,y) = \frac{x}{a} + \frac{y}{b}$ on $x^2 + y^2 = 1$.
   (b) Find the extreme values of $f(x,y) = x^2 + y^2$ on $\frac{x}{a} + \frac{y}{b} = 1$.
3. Find the nearest point from the origin to the curve of intersection of $x^2 - xy + y^2 - z^2 - 1 = 0$ and $x^2 + y^2 = 1$.

Nov. 9. Line integrals

1. Compute the line integrals $\int \boldsymbol{f} \cdot d\boldsymbol{\alpha}$
   (a) $\boldsymbol{f}(x,y) = (x^2 - 2xy, y^2 - 2xy), \boldsymbol{\alpha}(t) = (t, t^2), t \in [-1,1]$.
   (b) $\boldsymbol{f}(x,y,z) = (y^2 - z^2, 2yz, -x^2), \boldsymbol{\alpha}(t) = (t, t^2, t^3), t \in [-1,1]$.
   (c) $\boldsymbol{f}(x,y) = (y, -x), \boldsymbol{\alpha}(t) = (\cos t, \sin t), t \in [0,\pi]$ and $\boldsymbol{\beta}(t) = (-t, \sqrt{1-t^2}), t \in [-1,1]$.
2. A wire has a shape $x^2 + y^2 = a^2$, $a > 0$ with density $\varphi(x,y) = |x| + |y|$. Compute the mass.

Nov. 9. Gradients and line integrals

1. Show that the following vector fields $\boldsymbol{f}$ are not gradient. Find a closed path $\boldsymbol{\alpha}$ such that $\int \boldsymbol{f} \cdot d\boldsymbol{\alpha} \neq 0$.
   (a) $\boldsymbol{f}(x,y,z) = (y, x, x)$
   (b) $\boldsymbol{f}(x,y,z) = (xy, x^2 + 1, z^2)$
2. Show that, for a continuous function f, the vector field

$$\boldsymbol{f}(x,y) = \left(xf\left(\sqrt{x^2+y^2}\right), yf\left(\sqrt{x^2+y^2}\right)\right)$$

is a gradient.

3. Let $S = \{(x,y) \in \mathbb{R}^2, (x,y) \neq (0,0)\}, \boldsymbol{f}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$.

   (a) Show that $\partial_2 f_1 = \partial_1 f_2$.

   (b) For $\boldsymbol{\alpha}(t) = (\cos t, \sin t), t \in [0, 2\pi]$, show that $\int \boldsymbol{f} \cdot d\boldsymbol{\alpha} = 2\pi$, therefore, $\boldsymbol{f}$ is not a gradient on S.

Nov. 16. Potentials

1. Determine whether the following vector fields $\boldsymbol{f}$ are a gradient. If so, find a potential.

   (a) $\boldsymbol{f}(x,y) = \left(\frac{e^x}{e^x+y^2} \frac{2y}{e^x+y^2}\right)$ on $\mathbb{R}^2$.

   (b) $\boldsymbol{f}(x,y,z) = (2xyz + z^2 - 2y^2 + 1, x^2z - 4xy, x^2y + 2xy - 2)$ on $\mathbb{R}^3$.

   (c) $\boldsymbol{f}(x,y,z) = (2xz^3, x^2z^3, 3x^2yz^2)$.

2. Solve the following differential equations.

   (a) $\frac{dy}{dx} = -\frac{3x^2+6xy^2}{6x^2y+4y^3}$.

   (b) $y + 2xy' = 0$.

Nov. 16. Double integrals

1. Show that the function $f(x,y) = xy^3$ on $Q = [0,1] \times [0,1]$ is integrable.

2. The following functions are integrable. Compute $\iint_Q f(x,y)dxdy$.

   (a) $f(x,y) = xy(x+y), Q = [0,1] \times [0,1]$.

   (b) $f(x,y) = \sin(x+y), Q = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$.

   (c) $f(x,y) = y^{-3}e^{x/y}, Q = [0,1] \times [1,2]$.

Nov. 23. Double integrals

1. Compute the following integrals.

   (a) $\iint_Q (x \sin y - ye^x)dxdy$, $Q = [-1,1] \times [0, \frac{\pi}{2}]$.

   (b) $\iint_Q \sqrt{|y - x^2|}dxdy$, $Q = [-1,1] \times [0,2]$.

2. Compute the integral $\iint_S fdxdy$.

   (a) $f(x,y) = x\cos(x+y), S = \{(x,y) : 0 \leq x \leq \pi, 0 \leq y \leq x\}$.

   (b) $f(x,y) = x^2 - y^2, S = \{(x,y) : 0 \leq x \leq \pi, 0 \leq y \leq \sin x\}$.

   (c) $f(x,y) = 3x + y, S = \{(x,y) : 4x^2 + 9y^2 \leq 36, x \geq 0, y \geq 0\}$.

   (d) $f(x,y) = y + 2x + 20, S = \{(x,y) : x^2 + y^2 \leq 16\}$.

3. Write S as a type II region.

   (a) $S = \{(x,y) : 0 \leq x \leq 1, x^3 \leq y \leq x^2\}$.

   (b) $S = \{(x,y) : 1 \leq x \leq e, 0 \leq y \leq \log x\}$.

4. Find the centroid of S.

   (a) $S = \{(x,y) : 0 \leq x \leq \frac{\pi}{4}, \sin x \leq y \leq \cos x\}$.

   (b) $S = \{(x,y) : 1 \leq x \leq e, 0 \leq y \leq \log x\}$.

Nov. 30. Green's theorem

1. Compute the following line integrals.
   (a) $\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha}$, $\boldsymbol{f}(x,y) = (y^2, x)$ and C is the boundary of $[0,2] \times [0,2]$.
   (b) $\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha}$, $\boldsymbol{f}(x,y) = (3x - 3y, 4y + x)$ and $\boldsymbol{\alpha}(t) = (\cos t, \sin t), t \in [0, 2\pi]$.

2. With $S = \{(x,y) \in \mathbb{R}^2, (x,y) \neq (0,0)\}, \boldsymbol{f}(x,y) = \left(y + \frac{-y}{x^2+y^2}, 2x + \frac{x}{x^2+y^2}\right)$, show that $\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha}$, where $\boldsymbol{\alpha}(t) = (a \cos t, b \sin t), t \in [0, 2\pi]$ does not depend on $a, b > 0$.

Nov. 30. Change of coordinates

1. Find the corresponding region in the new coordinates.
   (a) $S = \{(x,y) : 0 \le x, 0 \le y, x + y \le 2\}, x = \frac{1}{2}(v - u), y = \frac{1}{2}(v + u)$.
   (b) $S = \{(x,y) : 0 \le x \le 1, x^2 + y^2 \le 1\}, x = r\cos\theta, y = r\sin\theta$.
   (c) $S = \{(x,y) : (x-a)^2 + y^2 \le a^2\}, x = r\cos\theta, y = r\sin\theta$.

2. Computer the integrals in the new coordinages.
   (a) $\iint_S e^{(y-x)/(y+x)} dxdy$, $S = \{(x,y) : 0 \le x, 0 \le y, x+y \le 2\}, x = \frac{1}{2}(v-u), y = \frac{1}{2}(v+u)$.
   (b) $\iint_S (x^2 + y^2) dxdy$, $S = \{(x,y) : (x-a)^2 + y^2 \le a^2\}, x = r\cos\theta, y = r\sin\theta$.

3. Compute the volume of the sphere $V = \{(x,y,z) : x^2 + y^2 + z^2 \le a^2\}$.

Dec. 14. Surface

1. Find a parametrization of the cylinder $\{(x,y,z) : x^2 + y^2 = a^2, 0 \le z \le 1\}$.

2. Compute $\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}$.
   (a) $\boldsymbol{r}(u,v) = (u + v, u - v, 4v^2)$.
   (b) $\boldsymbol{r}(u,v) = (a \sin u \cosh v, b \cos u \cosh v, c \sinh v)$.

3. Compute the area.
   (a) the intersection of $x + y + z = a, x^2 + y^2 \le a^2$.

Dec. 14. Surface integrals

1. Let $S : x^2 + y^2 + z^2 = 1, z \ge 0$ and $\boldsymbol{F}(x,y,z) = (x,y,0)$. Compute $\iint_S \boldsymbol{F} \cdot \boldsymbol{n} dS$ with the parametrization $z = \sqrt{1 - x^2 - y^2}$.

2. Let S be a triangle with vertices $(1,0,0), (0,1,0), (0,0,1)$ and $\boldsymbol{F}(x,y,z) = (x,y,z)$. Compute $\iint_S \boldsymbol{F} \cdot \boldsymbol{n} dS$, where $\boldsymbol{n}$ has positive z-component.

3. Compute curl and div.
   (a) $\boldsymbol{F}(x,y,z) = (2z - 3y, 3x - z, y - 2x)$
   (b) $\boldsymbol{F}(x,y,z) = (e^{xy}, \cos xy, \cos xz^2)$

Dec. 21. Stokes' theorem

1. Let C be the curve of the intersection $x^2 + y^2 + z^2 = a^2, x + y + z = 0$. Compute $\int \boldsymbol{F} \cdot d\boldsymbol{\alpha}$, where $\boldsymbol{F}(x, y, z) = (y, z, x)$.

2. Let $\boldsymbol{F}(x, y, z) = (e^{zy^2},\ e^{yx^2},\ e^{xz^2})$, C be the boundary of the square which has the vertices $(0, 0, 0), (1, 0, 0), (1, 1, 0), (0, 1, 0)$. Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is a parametrization of C going counterclockwise.

Dec. 21. Gauss' theorem

1. Let S be the surface of the unit cube $V = \{(x, y, z) : 0 \leq x, y, z \leq 1\}$, $\boldsymbol{n}$ be the outgoing unit vector on S, $\boldsymbol{F}(x, y, z) = (x^2, y^2, z^2)$. Compute $\iint_S \boldsymbol{F} \cdot \boldsymbol{n} dS$ and $\iiint_V \operatorname{div} \boldsymbol{F} dxdydz$.

2. Let $\boldsymbol{F}(x, y, z) = (x^3, y^3, z^3)$, $S : \{(x, y, z) : x^2 + y^2 + z^2 = a^2\}$, and $\boldsymbol{n}$ the outgoing normal unit vector on S at each point of S. Compute the surface integral $\iint\limits_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$.