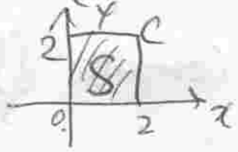
Exercises 11.22.

Use Green's theorem to evaluate $\int_C (y^2 dx + x dy)$, where

1(a) C is the boundary of the square (counter-clockwise)



Solution). Define the vector field $f(x,y) = (y^2, x)$. Or, $P(x,y) = y^2$, $Q(x,y) = x$.

With α parametrizing C, the given integral is $\int_C f \cdot d\alpha$,

and by Green's theorem, $\int_C f \cdot d\alpha = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy = \iint_S (1-2y) dx dy$

As the square S is of type I, $S = \{(x,y) : 0 \le x \le 2, 0 \le y \le 2\}$,

this can be evaluated by the iterated integral

$$\int_0^2 \int_0^2 (1-2y) dy\, dx = \int_0^2 [y - y^2]_0^2 dx = \int_0^2 (-2) dx = -4.$$

12.4. Compute $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$. (Recall that for $w_1 = (x_1, y_1, z_1)$, $w_2 = (x_2, y_2, z_2)$. $w_1 \times w_2 = (y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2)$)

7. $r(u,v) = (a \sin u \cosh v, b \cos u \cosh v, c \sinh v)$

Solution $\frac{\partial r}{\partial u} = (a \cos u \cosh v, -b \sin u \cosh v, 0)$.

$\frac{\partial r}{\partial v} = (a \sin u \sinh v, b \cos u \sinh v, c \cosh v)$.

$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = (-bc \sin u \cosh^2 v, -ac \cos u \cosh^2 v, ab(\cos^2 u + \sin^2 u) \sinh v \cosh v)$

$= (-bc \sin u \cosh^2 v, -ac \cos u \cosh^2 v, ab \sinh v \cosh v)$.

8. $r(u,v) = (u+v, u-v, 4v^2)$.

Solution $\frac{\partial r}{\partial u} = (1, 1, 0)$. $\frac{\partial r}{\partial v} = (1, -1, 8v)$.

$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = (8v, -8v, -2)$.

12.6. Compute the area. 2. the intersection of $x+y+z=a$, $x^2+y^2 \le a^2$.

Solution. The surface can be written, with $z = a-x-y$, as

$S = \{(x,y,z) : x^2+y^2 \le a^2, z = a-x-y\}$. Put $S_0 = \{(x,y) : x^2+y^2 \le a^2\}$

In other words, S is parametrized by $r(u,v) = (u, v, a-u-v)$, $u^2+v^2 \le a^2$.

$\frac{\partial r}{\partial u} = (1, 0, -1)$, $\frac{\partial r}{\partial u} = (0, 1, -1)$. $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = (1,1,1)$, $\left\|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right\| = \sqrt{3}$.

The area is $\iint_{S_0} \left\|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right\| dx dy = \sqrt{3} \iint_{S_0} dx dy = \pi a^2 \sqrt{3}$ (S_0 is a disk with radius a)

Exercise 12.10. $\underline{1(a)}$. Let $S: x^2+y^2+z^2=1, z\geq 0$ (hemisphere), and $\mathbb{F}(x,y,z)=(x,y,0)$ be a vector field.

Compute $\iint_S \mathbb{F}\cdot\mathbb{n}_1\, dS$ with the parametrization $z=\sqrt{1-x^2-y^2}$.

$\underline{\text{Solution}}$. With $S_0 := \{(x,y): x^2+y^2\leq 1\}$, S can be parametrized by
$\mathbb{r}(u,v)=(u,v,\sqrt{1-u^2-v^2})$, $(u,v)\in S_0$.

$$\frac{\partial \mathbb{r}}{\partial u}=\left(1,0,\frac{-u}{\sqrt{1-u^2-v^2}}\right),\ \frac{\partial \mathbb{r}}{\partial v}=\left(0,1,\frac{-v}{\sqrt{1-u^2-v^2}}\right),\ \frac{\partial \mathbb{r}}{\partial u}\times\frac{\partial \mathbb{r}}{\partial v}=\left(\frac{u}{\sqrt{1-u^2-v^2}},\frac{v}{\sqrt{1-u^2-v^2}},1\right)$$

As we know, $\iint_S \mathbb{F}\cdot\mathbb{n}_1 dS=\iint_{S_0}\mathbb{F}\cdot\left(\frac{\partial \mathbb{r}}{\partial u}\times\frac{\partial \mathbb{r}}{\partial v}\right)du\,dv$,

and $\mathbb{F}(\mathbb{r}(u,v))=(u,v,0)$. $\mathbb{F}\cdot\left(\frac{\partial \mathbb{r}}{\partial u}\times\frac{\partial \mathbb{r}}{\partial v}\right)=\frac{u^2+v^2}{\sqrt{1-u^2-v^2}}$. Using the polar coordinate,

$$\iint_{S_0}\frac{u^2+v^2}{\sqrt{1-u^2-v^2}}du\,dv=\int_0^{2\pi}\int_0^1\frac{r^2}{\sqrt{1-r^2}}\cdot r\,dr\,d\theta=\int_0^{2\pi}\int_0^1\frac{r^3}{\sqrt{1-r^2}}dr\,d\theta.$$

With $t=r^2$, $\frac{dt}{dr}=2r$, we get

$$\int_0^1\frac{r^3}{\sqrt{1-r^2}}dr=\frac{1}{2}\int_0^1\frac{t}{\sqrt{1-t}}dt=\frac{1}{2}\int_0^1\frac{(t-1)+1}{\sqrt{1-t}}dt$$

$$=\frac{1}{2}\int_0^1\left(-\sqrt{1-t}+\frac{1}{\sqrt{1-t}}\right)dt=\frac{1}{2}\left[-\frac{2}{3}(1-t)^{\frac{3}{2}}+2(1-t)^{\frac{1}{2}}\right]_0^1=\frac{1}{2}\cdot\frac{4}{3}=\frac{2}{3}.$$

Altogether, $\int_0^{2\pi}\int_0^1\frac{r^3}{\sqrt{1-r^2}}dr\,d\theta=\int_0^{2\pi}\frac{2}{3}d\theta=\frac{4}{3}\pi$.

12.13 $\underline{2.}$ $S: z=1-x^2-y^2, z\geq 0$, $\mathbb{F}(x,y,z)=(y,z,x)$.

Compute $\iint \text{curl}\,\mathbb{F}\cdot\mathbb{n}_1 dS$, using Stokes' theorem.

$\underline{\text{Solution}}$. The surface S can be parametrized by $\mathbb{r}(u,v)=(u,v,1-u^2-v^2)$,
$(u,v)\in S_0=\{(u,v): u^2+v^2\leq 1\}$. This has the boundary
$C := \{(x,y,z): x^2+y^2=1, z=0\}$. Parametrize it by $\alpha(t)=(\cos t,\sin t,0)$.

By Stokes' theorem, $\iint_S \text{curl}\,\mathbb{F}\cdot\mathbb{n}_1 dS=\int_C \mathbb{F}(\alpha(t))\cdot d\alpha$.

By noting that $\mathbb{F}(\alpha(t))=(\sin t,0,\cos t)$, $\alpha'(t)=(-\sin t,\cos t,0)$,

we have $\int_C \mathbb{F}(\alpha(t))\cdot d\alpha=\int_0^{2\pi}\mathbb{F}(\alpha(t))\cdot\alpha'(t)dt=\int_0^{2\pi}-\sin^2 t\,dt$.

Using $\sin^2 t=\frac{1}{2}(1-\cos 2t)$, $\int_0^{2\pi}-\sin^2 t\,dt=-\frac{1}{2}\int_0^{2\pi}(1-\cos 2t)dt=-\pi$.

12.13 5. Let C be the curve of the intersection $x^2+y^2+z^2=a,\ x+y+z=0$.
Prove that $\int_C \mathbb{F}\cdot d\alpha = \pi a^2\sqrt{3}$, where $\mathbb{F}(x,y,z)=(y,z,x)$.

Solution. From $z=-x-y$, $x^2+y^2+(-x-y)^2=a \iff 2x^2+2y^2+2xy=a^2,\ z=-x-y$
$\iff \frac{3}{2}(x+y)^2+\frac{1}{2}(x-y)^2=a,\ z=-x-y$, we see that C is an ellipse,
parametrized by $\mathbb{r}(u,v)=(u,v,-u-v)$, $\frac{3}{2}(u+v)^2+\frac{1}{2}(u-v)^2=a^2$.

C is the boundary of the surface $S: \mathbb{r}(u,v)=(u,v,-u-v),\ (u,v)\in S_0$,
where $S_0:=\{(u,v): \frac{3}{2}(u+v)^2+\frac{1}{2}(u-v)^2\le a^2\}$.
$\frac{\partial \mathbb{r}}{\partial u}=(1,0,-1)$, $\frac{\partial \mathbb{r}}{\partial v}=(0,1,-1)$. $\frac{\partial \mathbb{r}}{\partial u}\times\frac{\partial \mathbb{r}}{\partial v}=(1,1,1)$.
$\operatorname{curl}\mathbb{F}=(1,1,1)$. By Stokes' theorem,

$$\int_C \mathbb{F}\,d\alpha=\iint_S \mathbb{F}\cdot\mathbb{n}\,dS=\iint_{S_0}\mathbb{F}(\mathbb{r}(u,v))\cdot\frac{\partial \mathbb{r}}{\partial u}\times\frac{\partial \mathbb{r}}{\partial v}\,du\,dv=\iint_{S_0} 3\,du\,dv.$$

To perform the last integral, introduce $\begin{cases} s=\sqrt{\frac{3}{2}}(u+v)\\ t=\sqrt{\frac{1}{2}}(u-v)\end{cases} \iff \begin{cases} u=\frac{1}{\sqrt6}s+\frac{1}{\sqrt2}t\\ v=\frac{1}{\sqrt6}s-\frac{1}{\sqrt2}t\end{cases}$

$$J=\begin{vmatrix}\frac{\partial u}{\partial s} & \frac{\partial v}{\partial s}\\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t}\end{vmatrix}=\begin{vmatrix}\frac{1}{\sqrt6} & \frac{1}{\sqrt6}\\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2}\end{vmatrix}=-\frac{1}{\sqrt3}.$$
and $(u,v)\in S_0 \iff s^2+t^2\le a^2$.
$S_1=\{(s,t): s^2+t^2\le a^2\}$

$$\iint_{S_0} 3\,du\,dv=3\cdot\iint_{S_1}\left|-\frac{1}{\sqrt3}\right|ds\,dt=\sqrt3\cdot\pi a^2,$$
because S_1 is a disk with radius a.

12.21 1. Let S be the surface of the unit cube $V:=\{(x,y,z);\ 0\le x\le1,\ 0\le y\le1,\ 0\le z\le1\}$
$\mathbb{n}$ be the unit outer normal vector of S.
$\mathbb{F}(x,y,z)=(x^2,y^2,z^2)$ be a vector field.
Compute $\iint_S \mathbb{F}\cdot\mathbb{n}\,dS$ and $\iiint_V \operatorname{div}\mathbb{F}\,dx\,dy\,dz$.

Solution. — S consists of 6 squares. Let $S_{x,0}=\{(0,y,z): 0\le y\le1,\ 0\le z\le1\}$, $S_{x,1}=\{(1,y,z): 0\le y\le1,\ 0\le z\le1\}$
$\mathbb{n}$ on $S_{x,0}$ is $(-1,0,0)$. $\mathbb{F}=(0,y^2,z^2)$. so $\mathbb{F}\cdot\mathbb{n}=0$.
$\mathbb{n}$ on $S_{x,1}$ is $(1,0,0)$. $\mathbb{F}=(1,y^2,z^2)$. so $\mathbb{F}\cdot\mathbb{n}=1$. $\iint_{S_{x,1}} dS=\int_0^1\int_0^1 dy\,dz=1$.
Similarly, $\iint_{S_{y,0}}\mathbb{F}\cdot\mathbb{n}\,dS=0$, $\iint_{S_{y,1}}\mathbb{F}\cdot\mathbb{n}\,dS=1$, $\iint_{S_{z,0}}\mathbb{F}\cdot\mathbb{n}\,dS=0$, $\iint_{S_{z,1}}\mathbb{F}\cdot\mathbb{n}\,dS=1 \Rightarrow \iint_S \mathbb{F}\cdot\mathbb{n}\,dS=3$.

— $\operatorname{div}\mathbb{F}=2x+2y+2z$.

$$\iiint_V(2x+2y+2z)\,dx\,dy\,dz=\int_0^1\int_0^1\int_0^1(2x+2y+2z)\,dx\,dy\,dz=1+1+1=3$$