9.3 Exercises

1. Determine the solution of $4\frac{\partial f}{\partial x} + 3\frac{\partial f}{\partial y} = 0$ with $f(x,0) = \sin x$.

$(4e_1+3e_2)\cdot\nabla f = 0 \Rightarrow f$ is constant on $3x-4y=c$.

$\Rightarrow f(x,y) = g(3x-4y)$ for some g. $f(x,0) = g(3x) = \sin x$

$\Rightarrow g(x) = \sin\frac{x}{3}$, $f(x,y) = g(3x-4y) = \sin\frac{3x-4y}{3}$

9.5 1. Let $k>0$, $g(x,t) = \frac{1}{2}\frac{x}{\sqrt{kt}}$, $f(x,t) = \int_0^{g(x,t)} e^{-u^2}\,du$

(a) Show that $\frac{\partial f}{\partial x} = e^{-g^2}\frac{\partial g}{\partial x}$, $\frac{\partial f}{\partial t} = e^{-g^2}\frac{\partial g}{\partial t}$

$f(x,t) = F(g(x,t)) - F(0)$ where F is a primitive function of e^{-u^2}.

By the derivative of a composed function,

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x}\cdot e^{-g^2}, \quad \frac{\partial f}{\partial t} = \frac{\partial g}{\partial t}\cdot e^{-g^2}$$

(b). Show that $k\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t}$.

$$\frac{\partial g}{\partial x} = \frac{1}{2\sqrt{kt}}, \quad \frac{\partial^2 g}{\partial x^2} = 0, \quad \frac{\partial g}{\partial t} = -\frac{x}{4\sqrt{kt^3}}$$

$$k\frac{\partial^2 f}{\partial x^2} = k\left(\frac{\partial g}{\partial x}\right)^2\cdot(-2g)\cdot e^{-g^2} = k\frac{1}{4kt}\cdot\left(-\frac{x}{\sqrt{kt}}\right)\cdot e^{-g^2} = -\frac{x}{4\sqrt{kt^3}}\cdot e^{-g^2}$$

$\frac{\partial f}{\partial t} = -\frac{x}{4\sqrt{kt^3}}\cdot e^{-g^2}$, therefore, $\frac{\partial f}{\partial t} = k\frac{\partial^2 f}{\partial x^2}$.

2. Consider a function $f(x,y) = g(\sqrt{x^2+y^2})$

(a) Prove that, for $(x,y)\neq(0,0)$, $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{1}{r}g'(r) + g''(r)$, $r = \sqrt{x^2+y^2}$

$$\frac{\partial f}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}}g'(r). \quad \frac{\partial^2 f}{\partial x^2} = \frac{1}{r}g'(r) - \frac{2x^2}{2(x^2+y^2)^{\frac{3}{2}}}g'(r) + \frac{x^2}{r^2}g''(r)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{1}{r}g'(r) - \frac{y^2}{r^{\frac{3}{2}}}g'(r) + \frac{y^2}{r^2}g''(r).$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{2}{r}g'(r) - \frac{x^2+y^2}{r^{\frac{3}{2}}}g'(r) + \frac{x^2+y^2}{r^2}g''(r) = \frac{1}{r}g'(r) + g''(r).$$

(b) Assume that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ and prove that $f(x,y) = a\log(x^2+y^2) + b$

We have $\frac{1}{r}g'(r) + g''(r) = 0$. Put $h(r) = g'(r)$, then $\frac{1}{r}h(r) + h'(r) = 0$

$\Rightarrow h(r) = \frac{a'}{r}$, $a'\in\mathbb{R}$ (Thm 8.2). $\Rightarrow g(r) = a'\log r + b$

$f(x,y) = a\log\sqrt{x^2+y^2} + b = \frac{a'}{2}\log(x^2+y^2) + b$

Put $\frac{a'}{2} = a$.

9.8 Exercises

4. Consider two surfaces $2x^2+3y^2-z^2-25=0$, $x^2+y^2-z^2=0$.
The intersection C can be parametrized as $(X(z), Y(z), z)$ and passes the point $P=(\sqrt{7}, 3, 4)$.

(a) Find a unit tangent vector of C at P by implicit computations.

Put $F(x,y,z) = 2x^2+3y^2-z^2-25$, $G(x,y,z) = x^2+y^2-z^2$.

We use the formula $\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix}^{-1} \begin{pmatrix} -\frac{\partial F}{\partial z} \\ -\frac{\partial G}{\partial z} \end{pmatrix}$.

$\frac{\partial F}{\partial x} = 4x$, $\frac{\partial F}{\partial y} = 6y$, $\frac{\partial F}{\partial z} = -2z$, $\frac{\partial G}{\partial x} = 2x$, $\frac{\partial G}{\partial y} = 2y$, $\frac{\partial G}{\partial z} = -2z$.

$$\begin{pmatrix} X'(\sqrt{7},3,4) \\ Y'(\sqrt{7},3,4) \end{pmatrix} = \begin{pmatrix} 4\sqrt{7} & 18 \\ 2\sqrt{7} & 6 \end{pmatrix}^{-1} \begin{pmatrix} -(-8) \\ -(-8) \end{pmatrix} = \frac{1}{-12\sqrt{7}} \begin{pmatrix} 6 & -18 \\ -2\sqrt{7} & 4\sqrt{7} \end{pmatrix} \begin{pmatrix} 8 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{8}{\sqrt{7}} \\ -\frac{4}{3} \end{pmatrix}$$

(b) We have, from $3G-F=0$, $x^2+25=2z^2$, or, $x=\sqrt{2z^2-25}$

$F-2G=0$, $y^2=25-z^2$, or, $y=\sqrt{25-z^2}$.

$$X'(z) = \frac{4z}{2\sqrt{2z^2-25}} = \frac{2z}{\sqrt{2z^2-25}}, \quad Y'(z) = \frac{-2z}{2\sqrt{25-z^2}} = -\frac{z}{\sqrt{25-z^2}}$$

At P, $\begin{pmatrix} X' \\ Y' \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{8}{\sqrt{7}} \\ -\frac{4}{3} \\ 1 \end{pmatrix}$

9. The equation $x+z+(y+z)^2=6$ defines $z=f(x,y)$ implicitly.
Compute $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}$ in terms of x, y, z.

Put $F(x,y,z) = x+z+(y+z)^2-6$. $\frac{\partial F}{\partial x} = 1$, $\frac{\partial F}{\partial y} = 2(y+z)$, $\frac{\partial F}{\partial z} = 1+2(y+z)$.

$$\frac{\partial f}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{1}{1+2(y+z)}, \quad \frac{\partial f}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{2(y+z)}{1+2(y+z)}$$

$$\frac{\partial f}{\partial y} = -\frac{2(y+f(x,y))}{1+2(y+f(x,y))}, \quad \frac{\partial^2 f}{\partial x \partial y} = -\frac{2\cdot\left(-\frac{1}{1+2(y+z)}\right)\cdot(1+2(y+z)) - 2(y+z)\cdot 2\,\frac{-1}{1+2(y+z)}}{(1+2(y+f(x,y)))^2}$$

$$= \frac{2}{(1+2(y+z))^3}$$