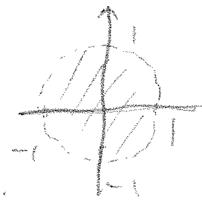
Exercises 8.3

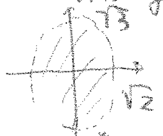
2. Are the following sets open? $(x,y)\in\mathbb{R}^2$

(a) $x^2+y^2<1$.

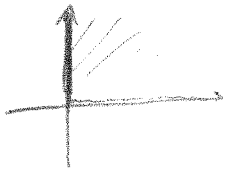
<u>Yes</u> Take an arbitrary (x_1,y_1) s.t. $x_1^2+y_1^2<1$.
Put $r_1=\sqrt{x_1^2+y_1^2}$. Then $B((x_1,y_1),1-r_1)$ is contained in $x^2+y^2<1$ by the triangle inequality.



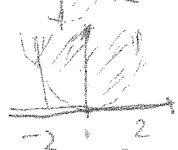
(b) $3x^2+2y^2<6$. <u>Yes</u>



(d) $x\geq 0$, $y>0$. <u>No.</u>



(e) $y>x^2$, $|x|<2$. <u>Yes.</u>



8.5 Determine where the following functions are defined and are continuous

1(a) $f(x,y)=x^4+y^4-4x^2y^2$ $\quad$ <u>$(x,y)\in\mathbb{R}^2$</u>

By putting $g(x,y)=x$, $h(x,y)=y$, $f(x,y)=g(x,y)^4+h(x,y)^4-4g(x,y)^2h(x,y)^2$ and one can apply Theorem 8.1. and Example 1.

(b) $f(x,y)=\log(x^2+y^2)$ $\quad$ $x^2+y^2>0$.

By putting $g_1(x,y)=x$, $g_2(x,y)=y$, $h(z)=\log z$, $f(x,y)=h(g_1(x,y)^2+g_2(x,y)^2)$ and one can apply Theorem 8.1, Example 1 and Theorem 8.2.

3 $f(x,y)=\dfrac{x-y}{x+y}$ if $x+y\neq 1$. f is discontinuous at $(0,0)$.

$\lim\limits_{x\to0} f(x,y)=-1$, hence $\lim\limits_{y\to0}\left[\lim\limits_{x\to0} f(x,y)\right]=-1$, while

$\lim\limits_{y\to0} f(x,y)=1$, hence $\lim\limits_{x\to0}\left[\lim\limits_{y\to0} f(x,y)\right]=1$.

8.14 Find the gradient $\nabla f=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)$, or $\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)$.

1. (a) $f(x,y)=x^2+y^2\sin(xy)$ $\quad \frac{\partial f}{\partial x}=2x+y^3\cos(xy)$, $\frac{\partial f}{\partial y}=2y\sin(xy)+xy^2\cos(xy)$.

(b) $f(x,y)=e^x\cos y$ $\quad \frac{\partial f}{\partial x}=e^x\cos y$. $\frac{\partial f}{\partial y}=-e^x\sin y$

(c) $f(x,y,z)=x^2y^3z^4$ $\quad \frac{\partial f}{\partial x}=2xy^3z^4$, $\frac{\partial f}{\partial y}=3x^2y^2z^4$, $\frac{\partial f}{\partial z}=4x^2y^3z^3$

2. Evaluate the directional derivative.

(a) $f(x,y,z)=x^2+2y^2+3z^2$ at $(1,1,0)$ in the direction $(1,-1,2)$

$\nabla f=(2x,4y,6z)$. $\nabla f(1,1,0)\cdot(1,-1,2)=(2,4,0)\cdot(1,-1,2)=-2$.

8.9 Check that $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ for the following function.

10 $f(x,y)=x^4+y^4-4x^2y^2$

$\frac{\partial f}{\partial y}=4y^3-8x^2y$. $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=-16xy$

$\frac{\partial f}{\partial x}=4x^3-8xy^2$. $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=-16xy$.